Numerical simulation of tsunami and its application

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Goals

- To understand the basic tsunami physics, modeling and its limitations

- To simulate a tsunami by yourself and check the validation of its results through simulation exercises using a tsunami code **(Tohoku University's Numerical Analysis Model for Investigation).**

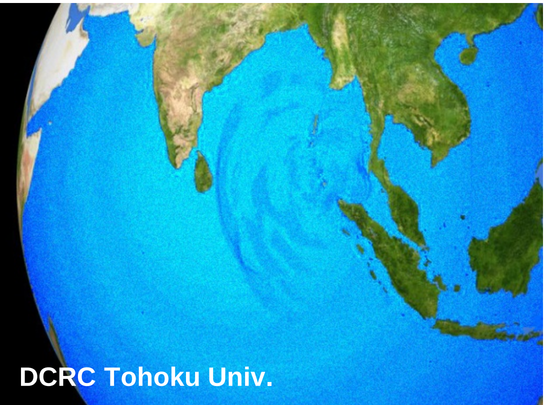
Contents

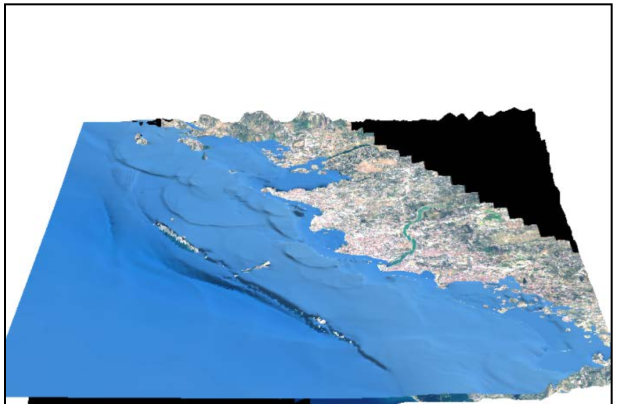
Introduction

Tsunami simulation is widely used for hazard assessments.

- Tsunami prediction
- Tsunami countermeasure
- Warning systems

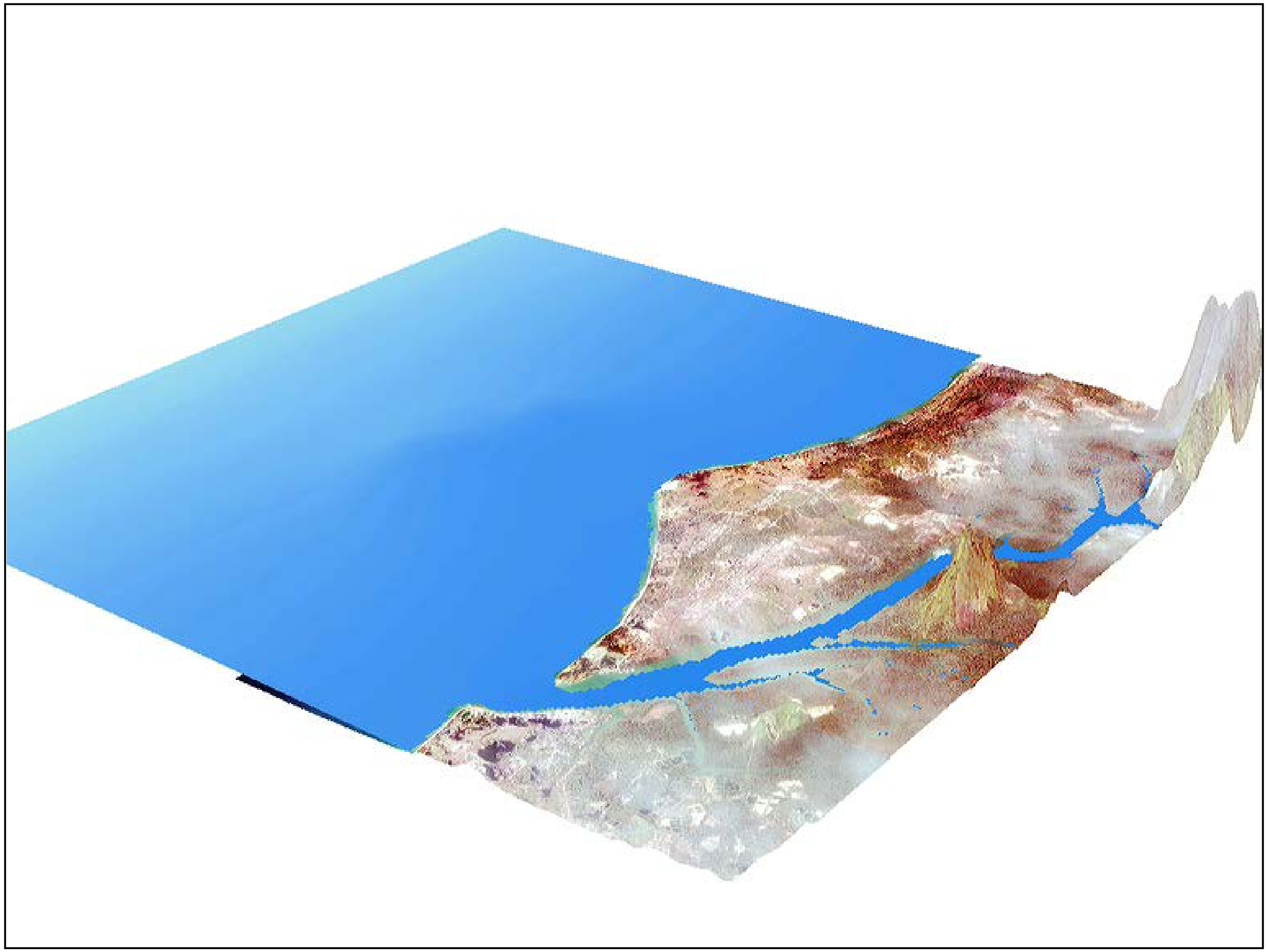
<u>2004 Indian Ocean tsunami</u>





Modeling of the 2004 tsunami

Namkem, Thailand



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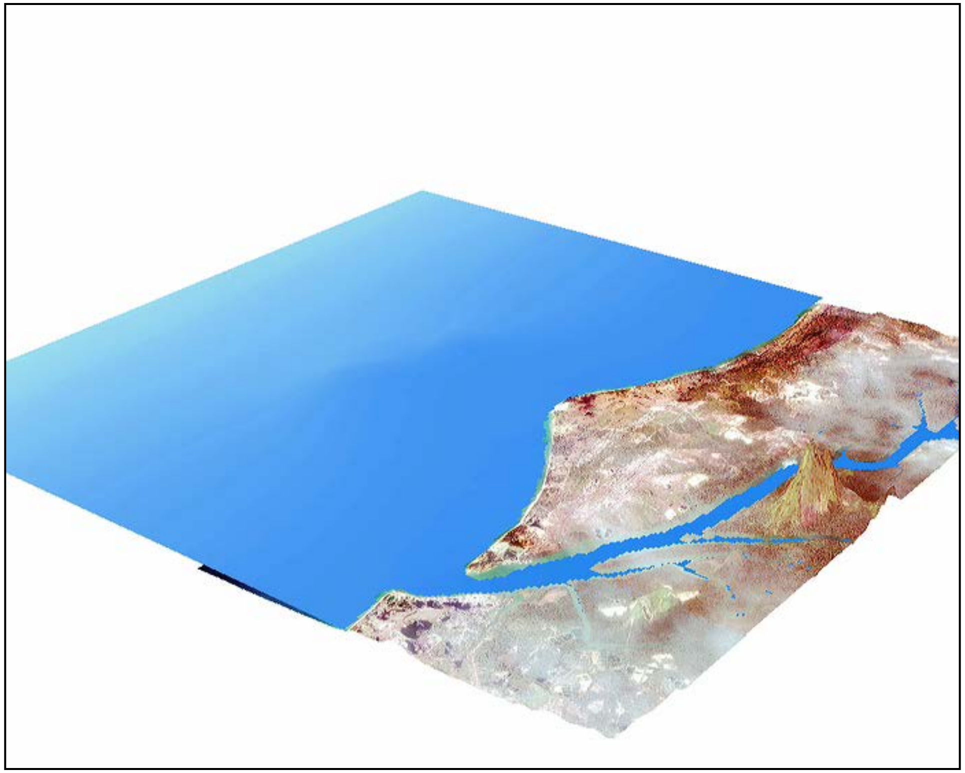
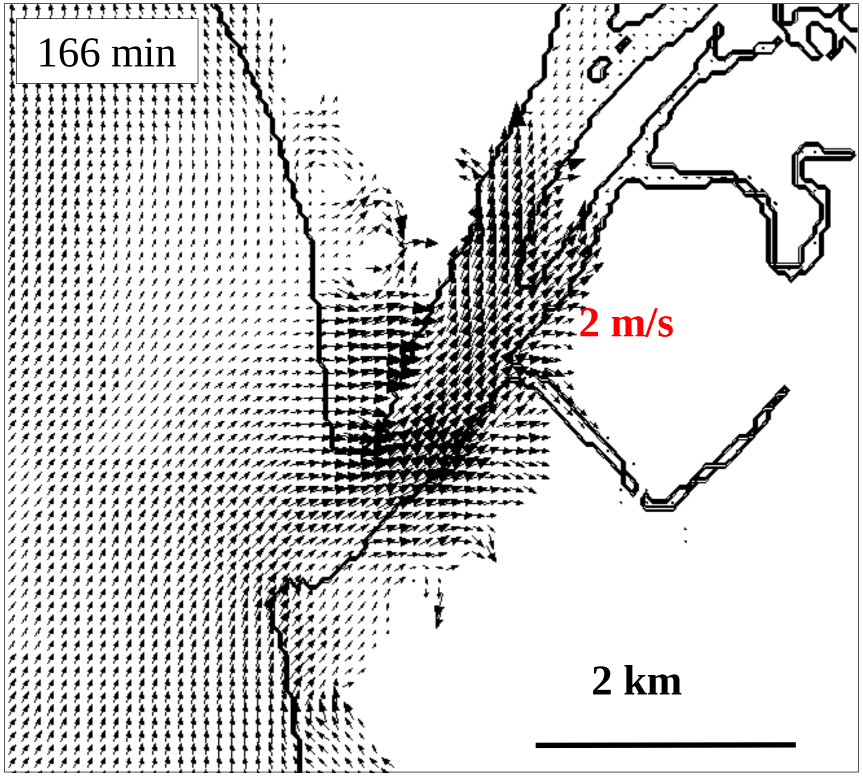


Tsunami current directions and velocity.

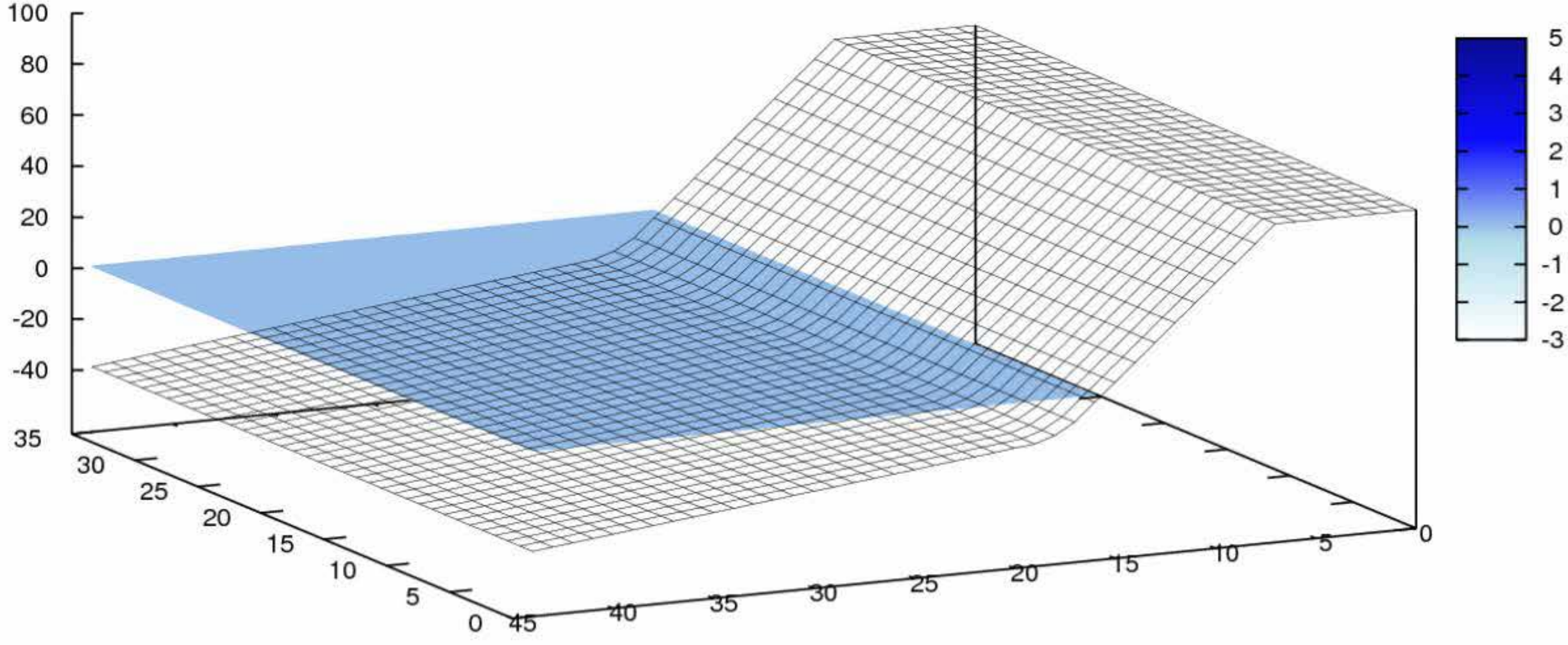


We can quantitatively estimate the impact of a tsunami flow from a numerical modeling

In today's lecture, I am going to explain tsunami modeling from fundamental basis to its application.

Advanced development model

Landslide-induced tsunami



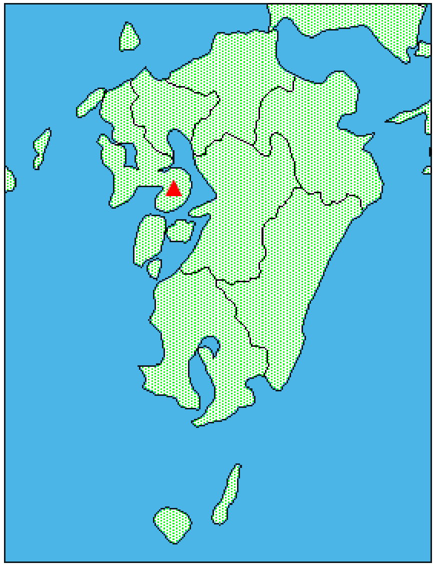
Example of landslide-induced tsunami

<u>島原大変肥後迷惑</u>

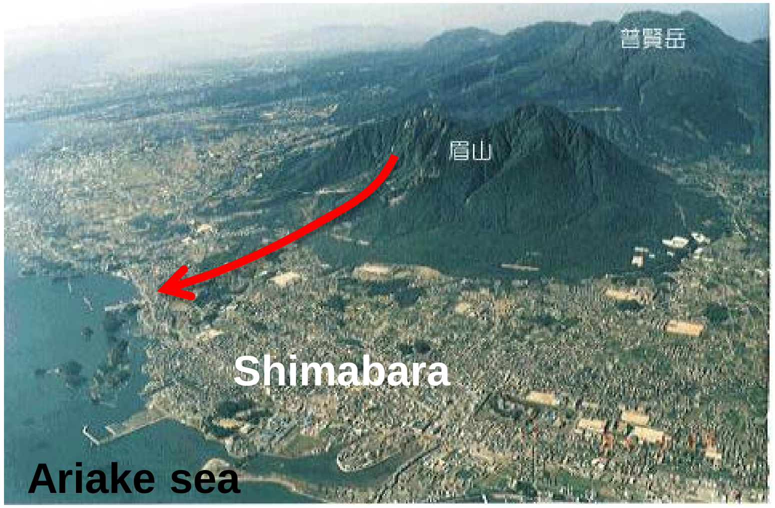
・ **1792 Unzen erupted**

・ **Mayuyama was collapsed with the large earthquake and then, the landslide flowed into Ariake-sea accompanying a large tsunami**

・ **The landslide and tsunami killed 15,000 people in Shimabara and Higo**



JMA website



Kyushu University Web Science Museum http://museum.sci.kyushu-u.ac.jp/Museum/Museum-e/Part2-e/taihen-e/taihen-e.htm

(C) Google earth

Outline of tsunami modeling

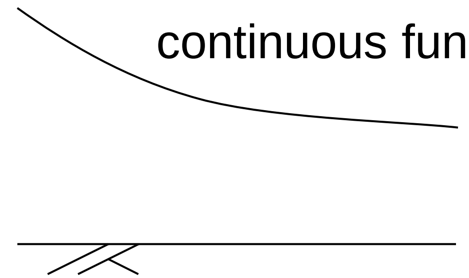
1. Derivation of governing equations

Differential equation

$$\frac{\partial \eta}{\partial t}+\frac{\partial M}{\partial x}=0$$

$$\frac{\partial M}{\partial t}+gh\frac{\partial \eta}{\partial x}=0$$

continuous function



2. Discretization of governing equations

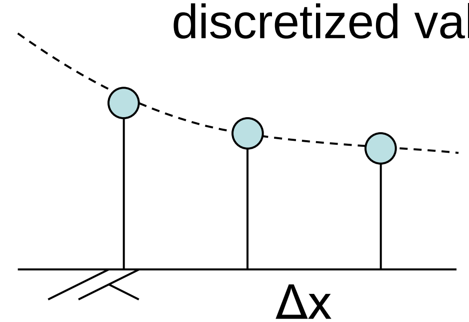
The differential equations could not be computed directly.

Finite difference method

$$\frac{\eta^{k+1}-\eta^{k}}{\Delta t}+\frac{M_{i+1/2}-M_{i-1/2}}{\Delta x}=0$$

$$\frac{M^{k+1/2}-M^{k-1/2}}{\Delta t}+gh_i\frac{\eta_{i+1}-\eta_i}{\Delta x}=0$$

discretized value



3. Programming using fortran

Outline of tsunami modeling

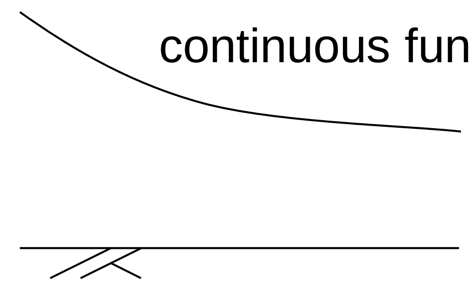
1. Derivation of governing equations

Differential equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0$$

$$\frac{\partial M}{\partial t} + gh\frac{\partial \eta}{\partial x} = 0$$

continuous function



2. Discretization of governing equations

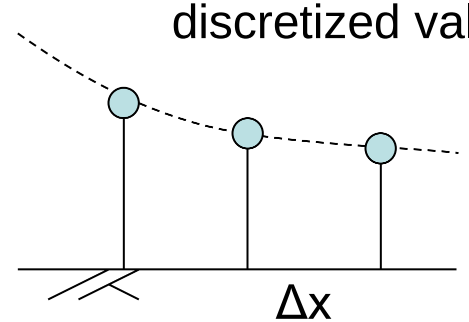
The differential equations could not be computed directly.

Finite difference method

$$\frac{\eta^{k+1} - \eta^{k}}{\Delta t} + \frac{M_{i+1/2} - M_{i-1/2}}{\Delta x} = 0$$

$$\frac{M^{k+1/2} - M^{k-1/2}}{\Delta t} + gh_i\frac{\eta_{i+1} - \eta_i}{\Delta x} = 0$$

discretized value

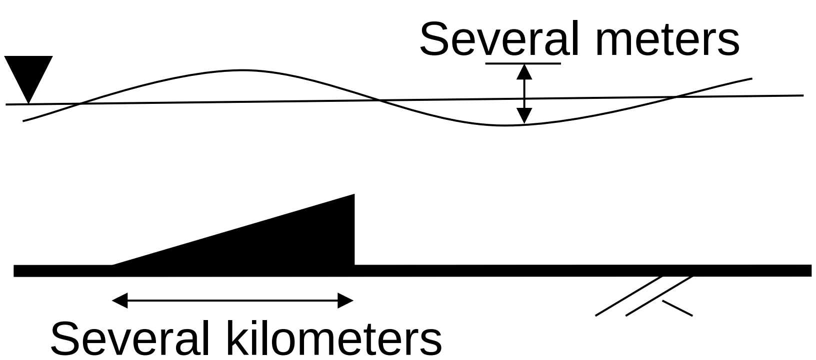


3. Programming using fortran

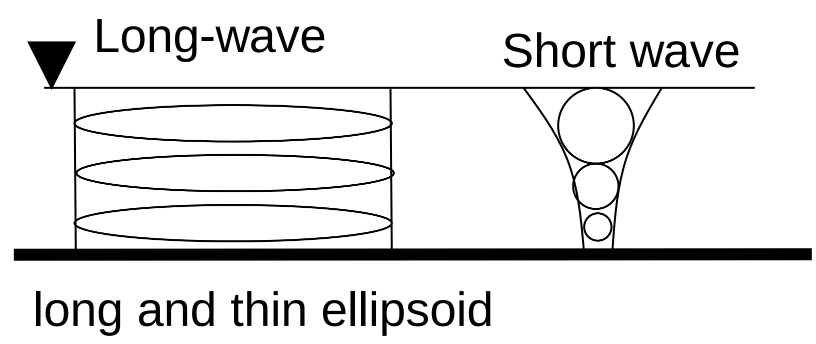
1. Differential equation (Long-wave approximation)

■ Long-wave approximation

1. Long wave (wave height << wave length)
2. Vertical acceleration of water particle $\fallingdotseq 0$
3. The water pressure is gravitational pressure



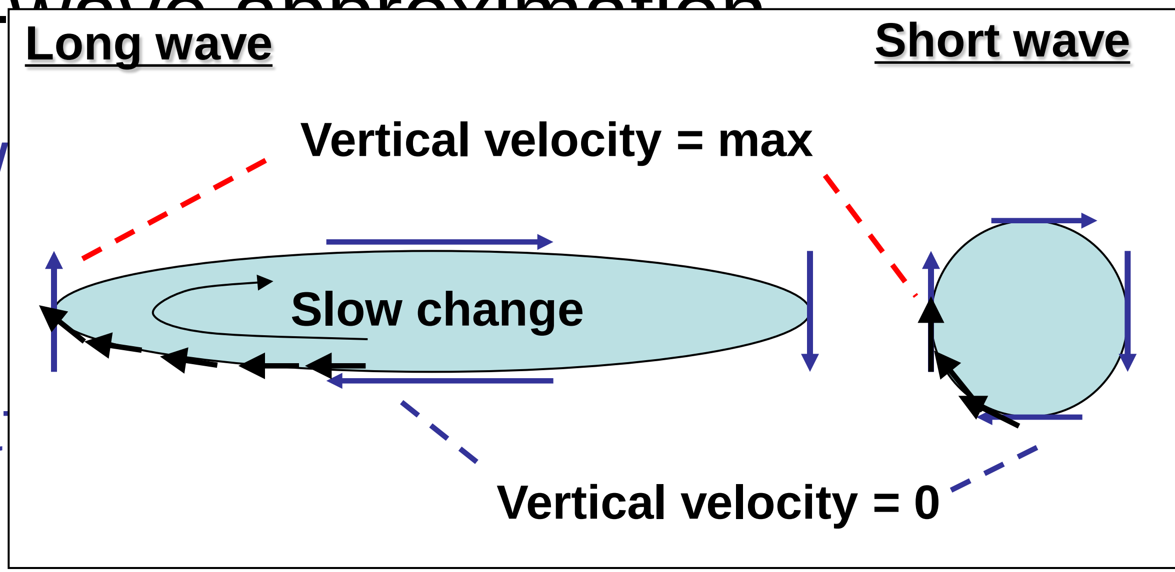
(Movement of water particle)



1. Differential equation (Long-wave approximation)

- Long-wave approximation

1. Long w
2. Vertical
3. The wa ... e



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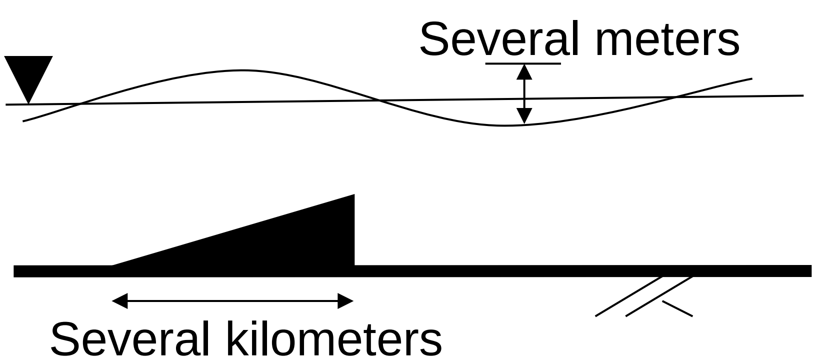

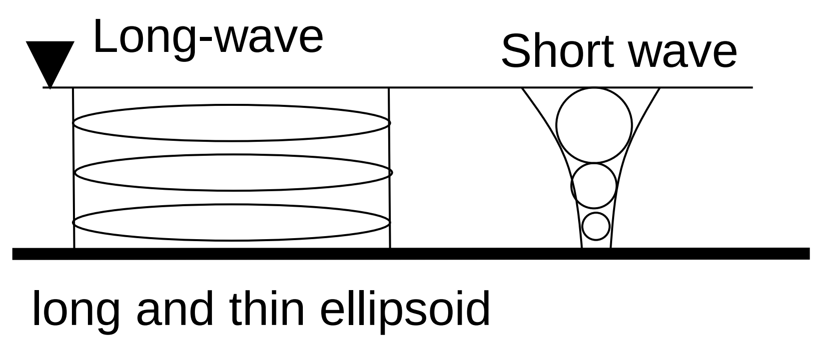

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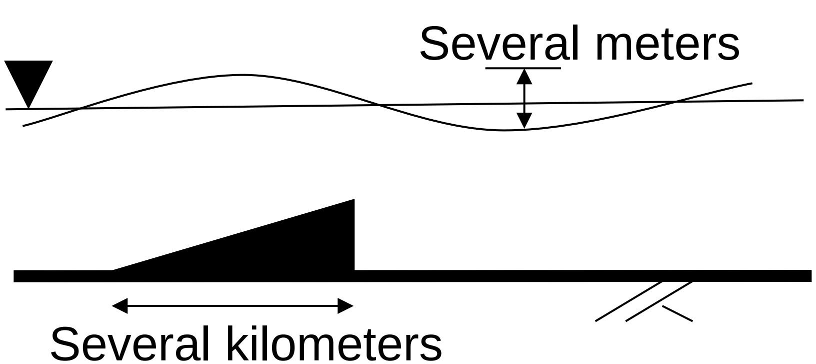
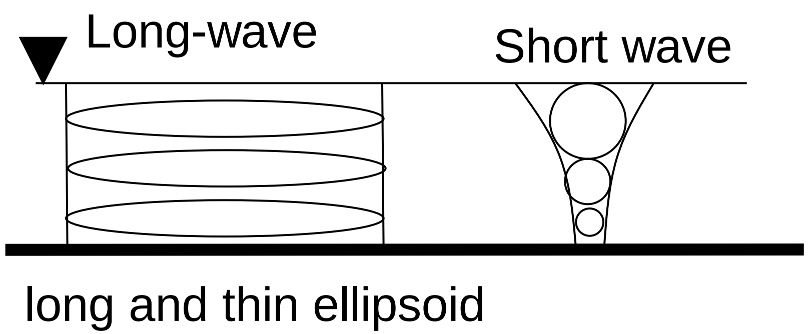
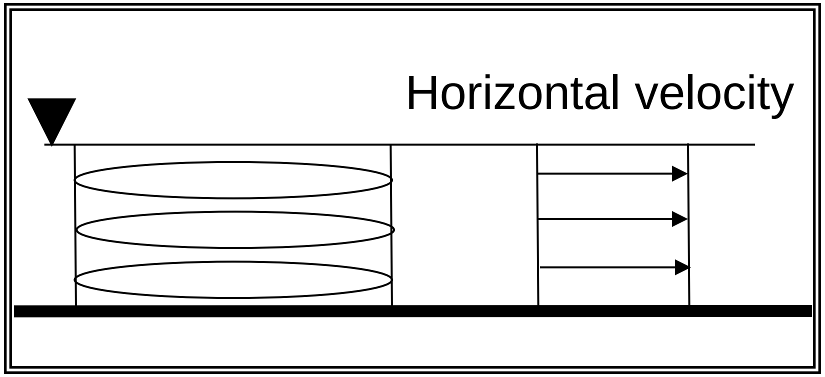


(Movement of water particle)



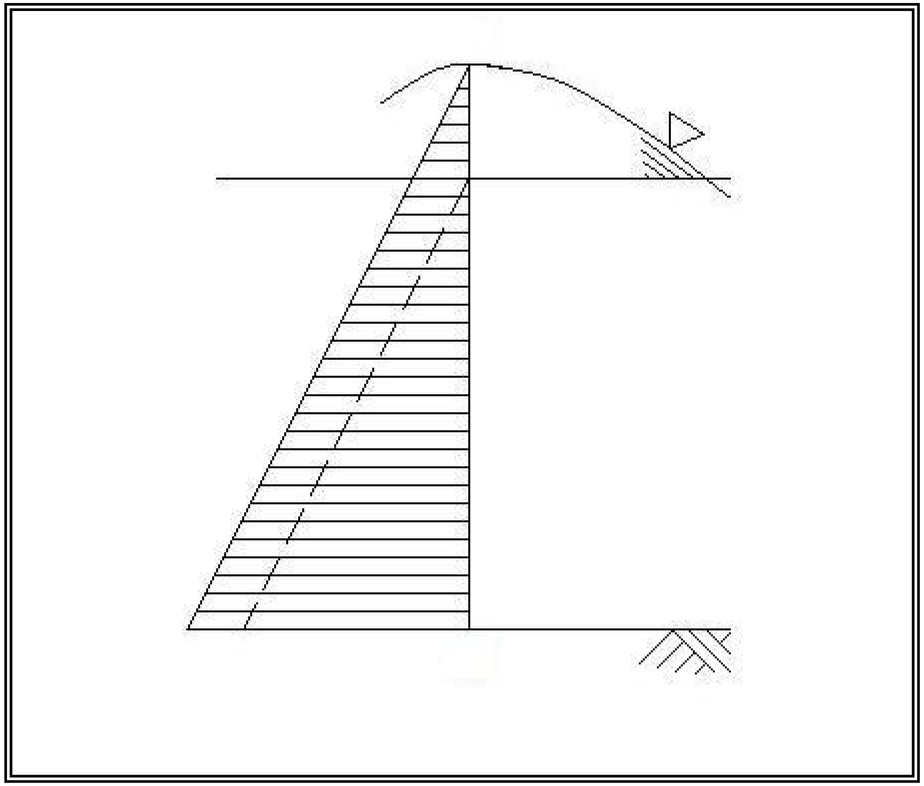
1. Differential equation (Long-wave approximation)

Movement of water particle



Horizontal velocity is vertically constant

Vertical distribution of wave pressure



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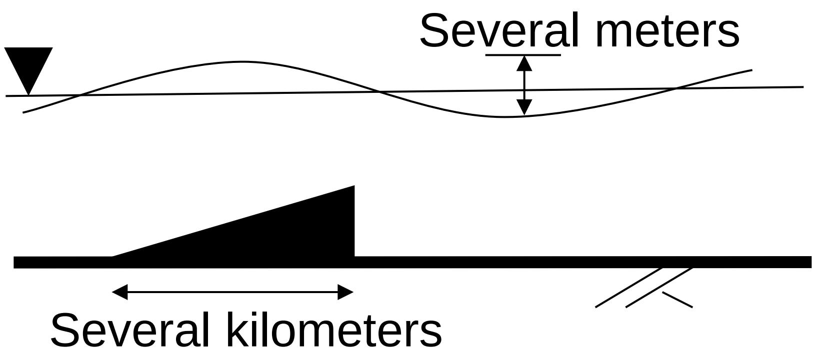
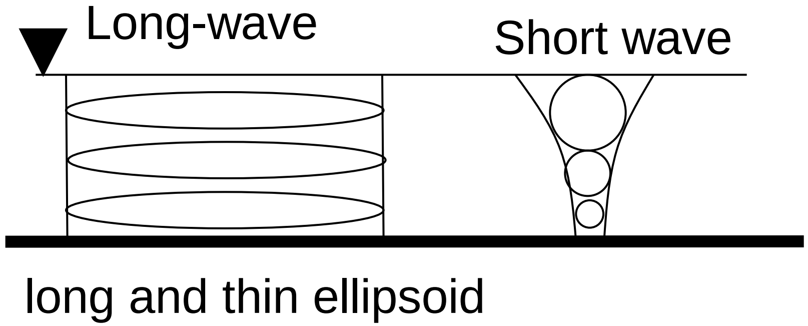


(Movement of water particle)

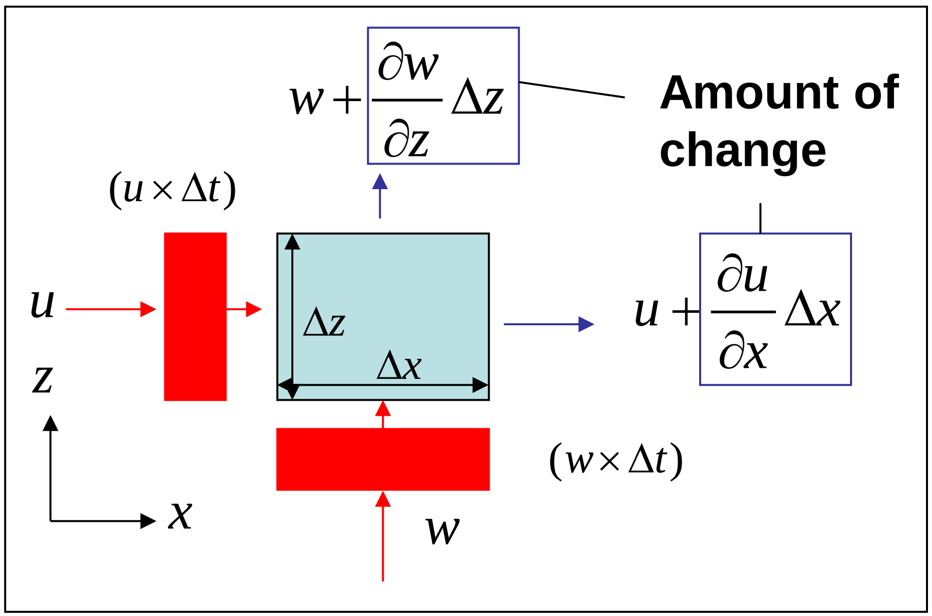


Governing equations (Differential equations)

1. Differential equation (Governing equations)

- The continuity equation of incompressible fluid

<u>Law of conservation of mass</u>



<u>Volume of inlet flow</u>

$$\rho\times\left\{\underbrace{(\Delta z\times u\times\Delta t)}_{\text{Volume from x-direction}}+\underbrace{(w\times\Delta x\times\Delta t)}_{\text{Volume from z-direction}}\right\}$$

<u>Volume of outlet flow</u>

$$\rho\left\{\underbrace{(u+\frac{\partial u}{\partial x}\Delta x)\Delta z\Delta t}_{\text{Volume from x-direction}}+\underbrace{(w+\frac{\partial w}{\partial z}\Delta z)\Delta x\Delta t}_{\text{Volume from z-direction}}\right\}$$

in out

$$\rho \cancel{u\Delta z\Delta t}+\rho \cancel{w\Delta x\Delta t}=\rho(\cancel{u}+\boxed{\frac{\partial u}{\partial x}}\Delta x)\cancel{\Delta z\Delta t}+(\cancel{w}+\boxed{\frac{\partial w}{\partial z}}\Delta z)\cancel{\Delta x\Delta t} \Rightarrow \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

1. Differential equation (Governing equations)

- The momentum equations of incompressible fluid

<u>Momentum condition:</u> Navier-stokes equation

Newton's second law

$$F = ma$$

<u>Horizontal direction</u>

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \boxed{\nu\left(\frac{\partial^2 u}{\partial z^2}\right)}$$

Horizontal acceleration of water particle

Pressure term

Viscosity term

<u>Vertical direction</u>

$$\boxed{\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}} = g - \boxed{\frac{1}{\rho}\frac{\partial p}{\partial z}} + \boxed{\nu\left(\frac{\partial^2 w}{\partial z^2}\right)}$$

Vertical acceleration of water particle

Pressure term

Viscosity term

u :Horizontal velocity

w :Vertical velocity

p :Water pressure

ν :Viscosity coefficient

g :Gravity acceleration

1. Differential equation (Governing equations)

Governing equation

- Continuity equation
- Navie-stokes equation

Direct numerical simulation

(Vof, smac, etc.)

Computationally expensive

Long-wave approximation

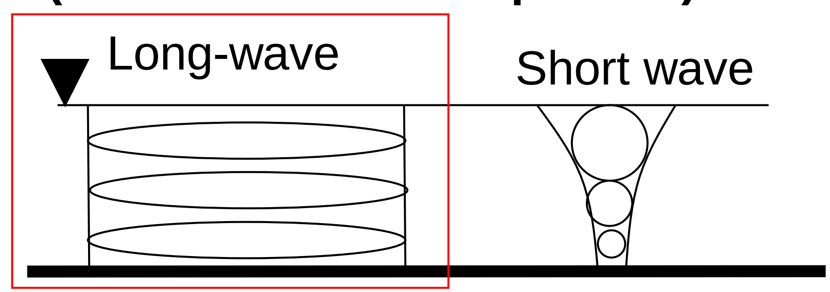
$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=Fx-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=Fz-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial z^2}\right)$$

These are fundamental equations for not only tsunami but also all incompressible fluid.

(Movement of water particle)

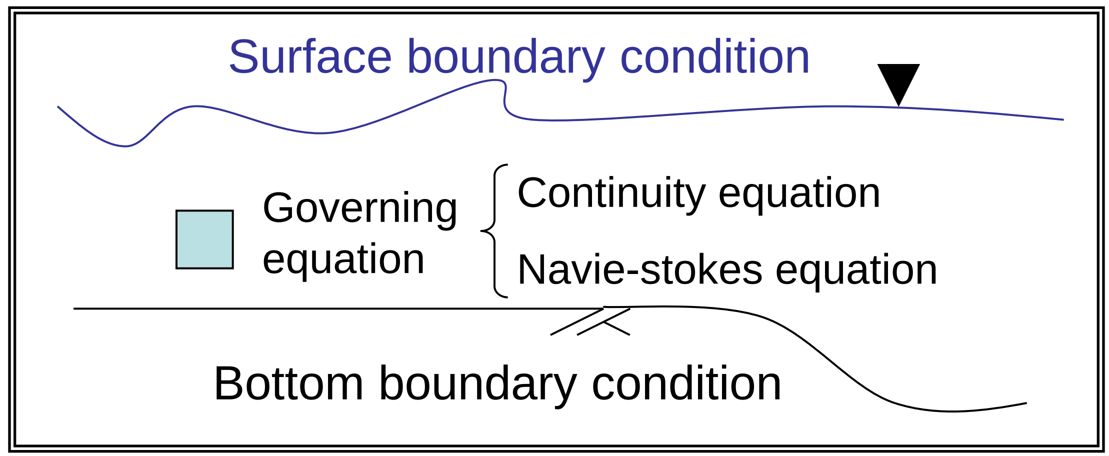


The long-wave equation

Flow for long-wave model

Actually long-wave model is derived from solving boundary value problems

Governing equations ← ① **Boundary conditions**

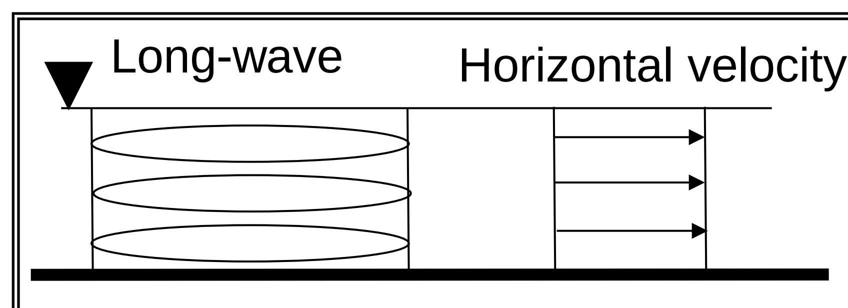


Long-wave approximation

(small vertical velocity)

↓

② **Integration of governing equations in the vertical direction**



And then, I integrate governing equations in the vertical direction to remove vertical component from governing equation.

1. Differential equation
(① Boundary conditions)

- Surface boundary (Z=η)

  a) Dynamic boundary condition

$$p_{surface} = 0$$

  b) Kinematic boundary condition

$$\boxed{\left(\frac{\partial \eta}{\partial t} + u_{surface}\frac{\partial \eta}{\partial x}\right)} \times \Delta t = \boxed{w_{surface}} \times \Delta t$$

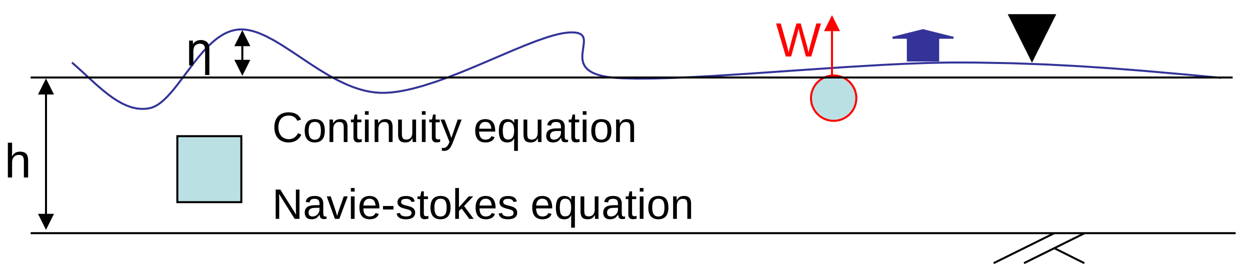
Rise velocity of water surface

Vertical velocity of water particle at surface

η : surface elevation (m)

u : Horizontal velocity (m/s)

w : vertical velocity (m/s)



1. Differential equation (① Boundary conditions)

- Bottom boundary (Z= -h)

c) Kinematic boundary condition

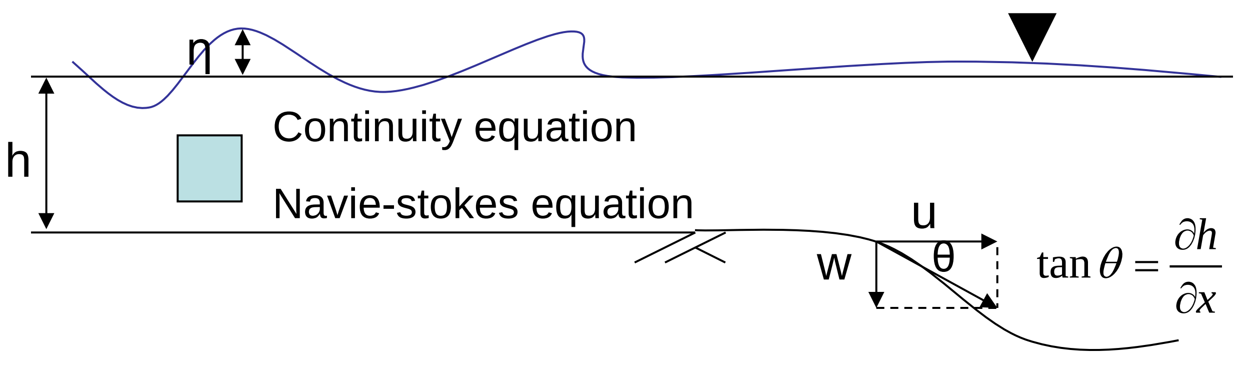
$$\tan\theta = \frac{w_b}{u_b} = -\frac{\partial h}{\partial x}$$

h : water depth (m)

u : Horizontal velocity (m/s)

w : vertical velocity (m/s)

the ratio between vertical velocity and horizontal velocity at bottom is equal to the gradient of bottom



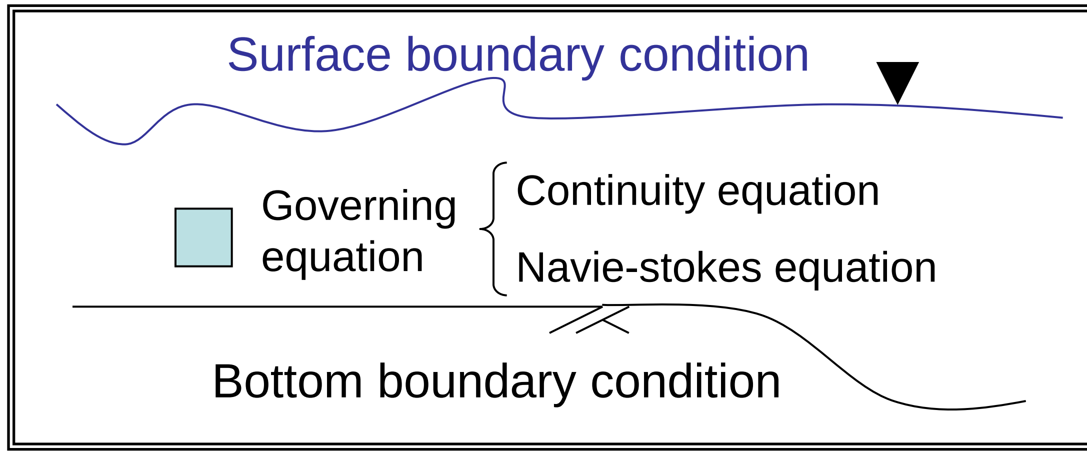
The bottom of water must flow along bottom bed.

Flow of long-wave approximation

①

Boundary equations

Governing equations

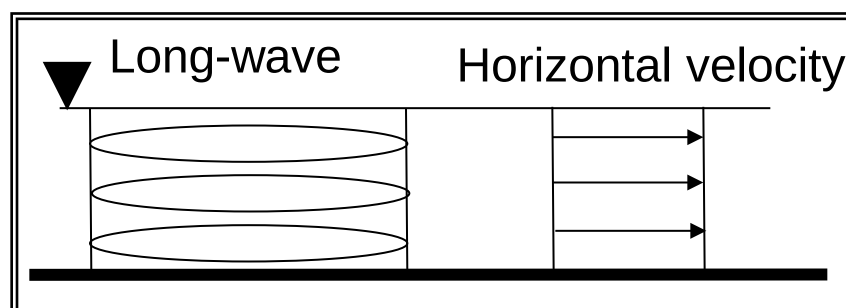


Long-wave approximation

(small vertical velocity)

②

Integration of governing equations in the vertical direction



I integrate governing equations in the vertical direction to remove vertical componet from governing equation.

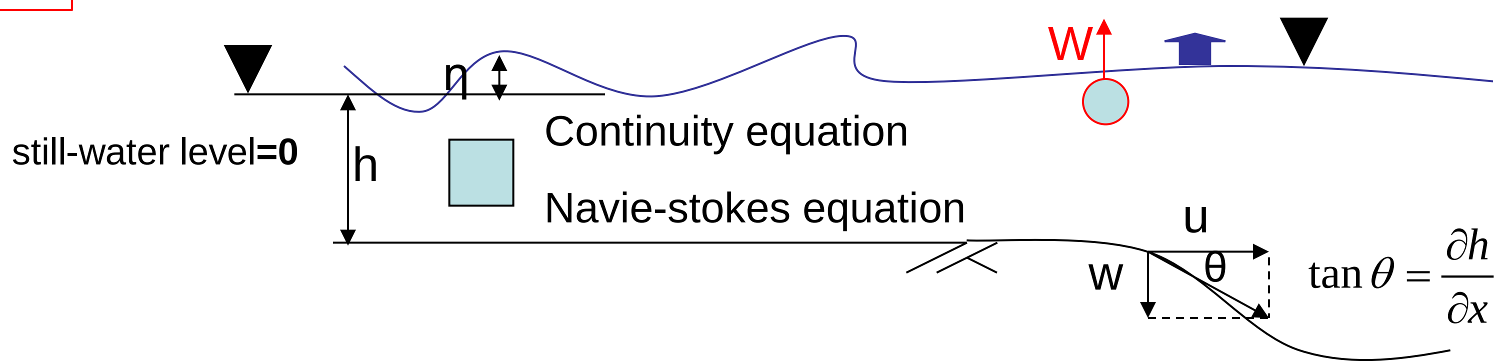
1. Differential equation
(② Integration of continuity equation)

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \longrightarrow \int_{bottom}^{Surface}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}dz \longrightarrow \int_{bottom}^{Surface}\frac{\partial u}{\partial x}dz+\int_{bottom}^{Surface}\frac{\partial w}{\partial z}dz$$

Integration in vertical direction

$$\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz+\boxed{\int_{-h}^{\eta}\frac{\partial w}{\partial z}dz}=$$



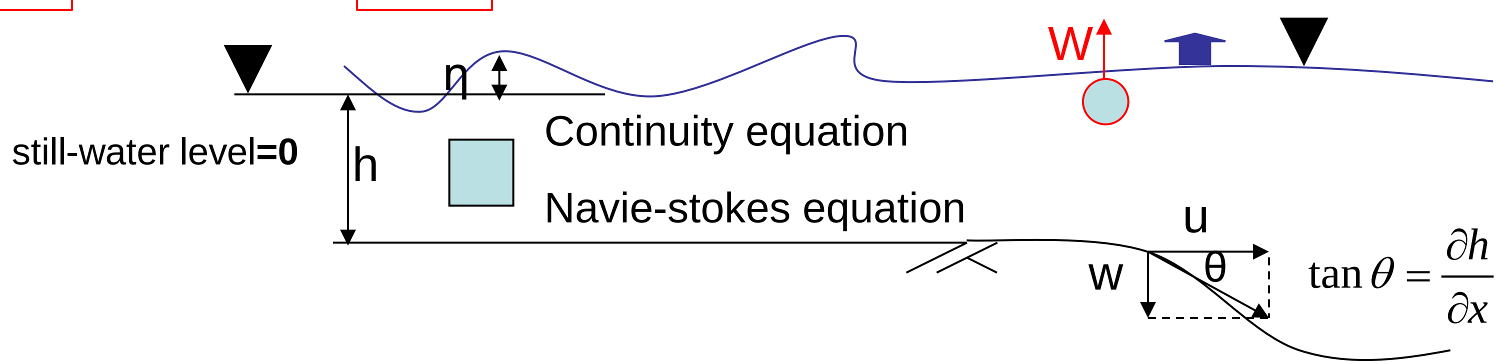
1. Differential equation
(② Integration of continuity equation)

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \longrightarrow \int_{bottom}^{Surface}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}dz \longrightarrow \int_{bottom}^{Surface}\frac{\partial u}{\partial x}dz+\int_{bottom}^{Surface}\frac{\partial w}{\partial z}dz$$

Integration in vertical direction

$$\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz+\boxed{\int_{-h}^{\eta}\frac{\partial w}{\partial z}dz}=\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz+\boxed{[w]_{-h}^{\eta}}=$$



1. Differential equation (② Integration of continuity equation)

Continuity equation

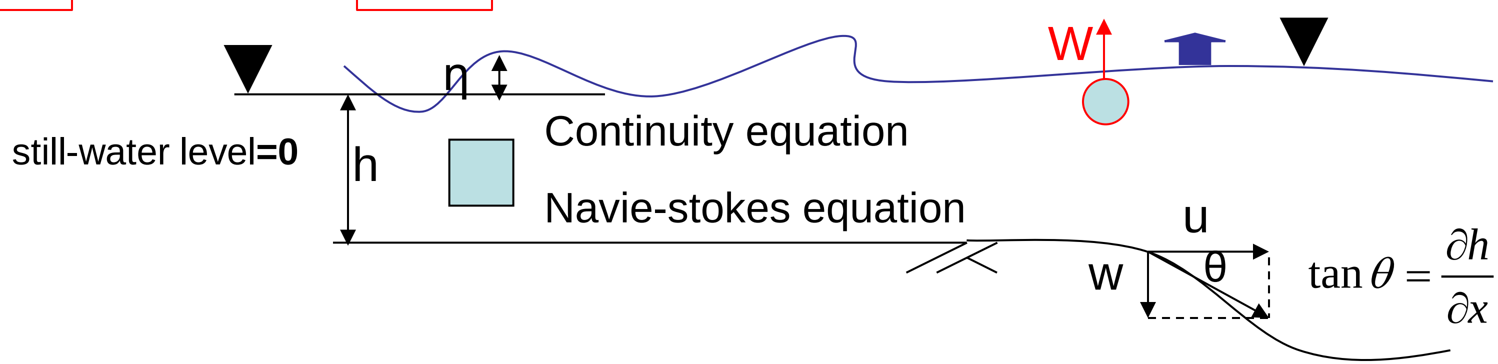
$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \longrightarrow \int_{bottom}^{Surface}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}dz \longrightarrow \int_{bottom}^{Surface}\frac{\partial u}{\partial x}dz+\int_{bottom}^{Surface}\frac{\partial w}{\partial z}dz$$

Integration in vertical direction

Vertical velocity at surface boundary

Vertical velocity at bottom boundary

$$\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz+\boxed{\int_{-h}^{\eta}\frac{\partial w}{\partial z}dz}=\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz+\boxed{[w]_{-h}^{\eta}}=\int_{-h}^{\eta}\frac{\partial u}{\partial x}dz+w(x,\boxed{\eta},t)-w(x,\boxed{-h},t)$$



$$\tan\theta=\frac{\partial h}{\partial x}$$

1. Differential equation
(② Integration of continuity equation)

Continuity equation

Surface boundary Bottom boundary

$$\int_{bottom}^{Surface} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} dz = \int_{-h}^{\eta} \frac{\partial u}{\partial x} dz + \boxed{w(x,\eta,t)} - \boxed{w(x,-h,t)}$$

$$= \boxed{\int_{-h}^{\eta} \frac{\partial u}{\partial x} dz} + \boxed{\left(\frac{\partial \eta_s}{\partial t} + \cancel{u_s \frac{\partial \eta_s}{\partial x}} \right)} - \boxed{\cancel{\left(-u_b \frac{\partial h}{\partial x} \right)}}$$

Boundary conditions

$$w_s = \left(\frac{\partial \eta_s}{\partial t} + u_s \frac{\partial \eta_s}{\partial x} \right)$$

$$\frac{w_b}{u_b} = -\frac{\partial h}{\partial x}$$

$$\int_{-h}^{\eta} \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz - \cancel{u_s \frac{\partial \eta}{\partial t}} - \cancel{u_b \frac{\partial h}{\partial t}}$$

Leibniz integral rule

$$\frac{\partial}{\partial x} \int_{\alpha(x)}^{\beta(x)} Q(x,y) dz =$$

$$\int_{\alpha(x)}^{\beta(x)} \frac{\partial}{\partial x} Q(x,y) dz + Q(x,\beta(x)) \frac{\partial \beta(x)}{\partial x} - Q(x,\alpha(x)) \frac{\partial \alpha(x)}{\partial t}$$

$$\boxed{\int_{bottom}^{Surface} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} dz = \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz + \frac{\partial \eta}{\partial t}}$$

1. Differential equation (② Integration of continuity equation)

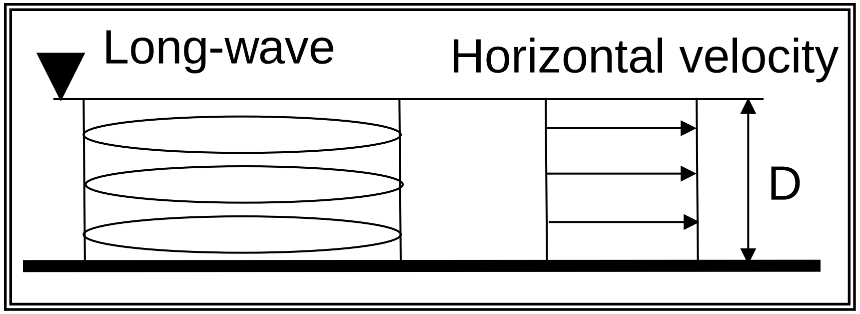
<u>From long-wave approximation, horizontal velocity is vertically constant.</u>

$$\frac{\partial}{\partial t}\int_{-h}^{\eta} u\,dz = \boxed{\frac{\partial}{\partial t}\left(\underline{uD}\right)}$$

Flux

$$M = u \times D$$

D :Depth from bottom to surface (=η+h)



Horizontal velocity is vertically constant

Continuity equation in long-wave model

$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0$$

η : surface elevation

M : Flux in the x-direction

1. Differential equation (② Integration of momentum equation)

Momentum equation in vertical direction

$$\cancel{\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}}=g-\frac{1}{\rho}\frac{\partial p}{\partial z}+\cancel{\nu\left(\frac{\partial^2 w}{\partial z^2}\right)}$$

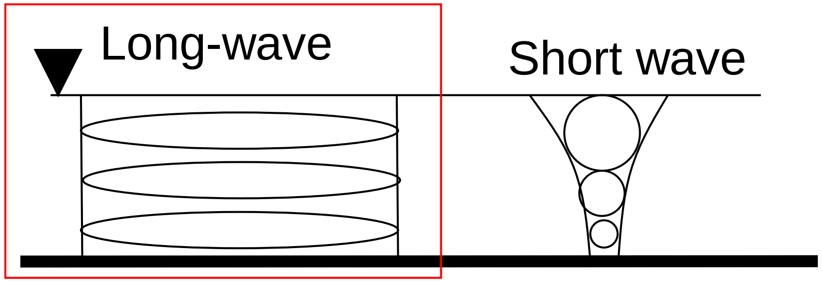
Vertical acceleration of water particle

Viscosity term

In the long wave condition, vertical acceleration of water particle is expected to be small

$$\longrightarrow \quad 0=g-\frac{1}{\rho}\frac{\partial p}{\partial z}$$

(Movement of water particle)



1. Differential equation (② Integration of momentum equation)

Indefinite Integration in vertical direction

$$0 = g - \frac{1}{\rho}\frac{\partial p}{\partial z} \longrightarrow 0 = \int\left(g - \frac{1}{\rho}\frac{\partial p}{\partial z}\right)dz \longrightarrow 0 = \underline{\rho\int g dz} - \underline{\int \frac{\partial p}{\partial z}dz}$$

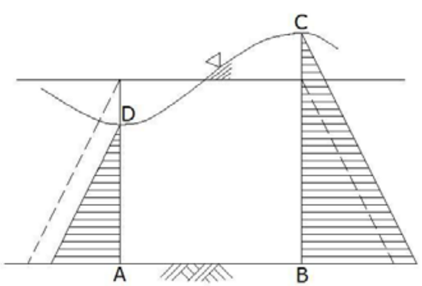
$$0 = \underline{\rho g z} - \underline{(p(x,z) + c)}$$ (C is a constant of integration)

From kinematic boundary condition $p_{surface} = 0$

$$p(\cancel{x,}\eta) = \rho g \eta - c \longrightarrow c = \rho g \eta$$

$$p(x,z) = \boxed{\rho g(\eta - z)}$$

Gravitational pressure



1. Differential equation
(② Integration of momentum equation)

Momentum equation in horizontal direction

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

$$\boxed{p=\rho g(\eta-z)}$$

$$\downarrow$$

$$\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial wu}{\partial z}=-g\frac{\partial \eta}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial z^2}\right)$$

$$\boxed{\frac{\partial u^2}{\partial x}+\frac{\partial wu}{\partial z}=u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}+\cancel{u\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)}}$$

Integration in vertical direction

$$\int_{bottom}^{surface}\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial wu}{\partial z}dz=\int_{bottom}^{surface}-g\frac{\partial \eta}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial z^2}\right)dz$$

1. Differential equation
(② Integration of momentum equation)

Integration in vertical direction

Leibniz integral rule

$$\frac{\partial}{\partial x}\int_{\alpha(x)}^{\beta(x)} Q(x,y)dz =$$
$$\int_{\alpha(x)}^{\beta(x)}\frac{\partial}{\partial x}Q(x,y)dz + Q(x,\beta(x))\frac{\partial \beta(x)}{\partial x} - Q(x,\alpha(x))\frac{\partial \alpha(x)}{\partial x}$$

(Left side terms)

$$\int_{bottom}^{surface}\boxed{\frac{\partial u}{\partial t}}+\boxed{\frac{\partial u^2}{\partial x}}+\boxed{\frac{\partial wu}{\partial z}}dz$$

$$\int_{bottom}^{surface}\frac{\partial u}{\partial t}dz = \frac{\partial}{\partial t}\int_{-h}^{\eta}udz - \cancel{u_{surface}\frac{\partial \eta}{\partial t}} - \cancel{u_{bottom}\frac{\partial h}{\partial t}}$$

$$\int_{bottom}^{surface}\frac{\partial u^2}{\partial x}dz = \frac{\partial}{\partial x}\int_{-h}^{\eta}u^2dz - \cancel{u_{surface}^2\frac{\partial \eta}{\partial x}} - \cancel{u_{bottom}^2\frac{\partial h}{\partial x}}$$

$$\int_{bottom}^{surface}\frac{\partial wu}{\partial z}dz = [uw]_{bottom}^{surface}$$

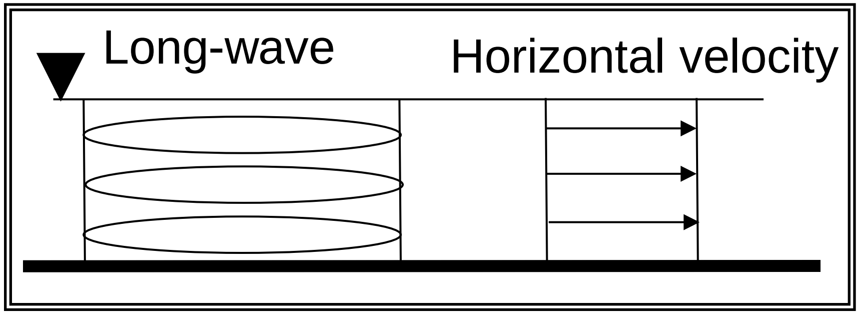
$$= u_{surface}\left(\cancel{\frac{\partial \eta}{\partial t}} + \cancel{u_{surface}\frac{\partial \eta}{\partial x}}\right) + u_{bottom}^2\cancel{\frac{\partial h}{\partial x}}$$

$$\left.\right\} = \frac{\partial}{\partial t}\int_{-h}^{\eta}udz + \frac{\partial}{\partial x}\int_{-h}^{\eta}u^2dz$$

1. Differential equation
(② Integration of momentum equation)

Integration in vertical direction

(Left side terms)



Horizontal velocity is vertically constant

$$\frac{\partial}{\partial t}\int_{-h}^{\eta} u\,dz = \boxed{\frac{\partial}{\partial t}(uD)}$$

$$\frac{\partial}{\partial t}\int_{-h}^{\eta} u^2\,dz = \boxed{\frac{\partial}{\partial x}\left(u^2 D\right)}$$

D :Depth from bottom to surface (=η+h)

$$\int_{bottom}^{surface} \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial wu}{\partial z}\,dz = \frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right)$$

1. Differential equation (② Integration of momentum equation)

Integration in vertical direction

$$\int_{bottom}^{surface} \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial wu}{\partial z} dz = \boxed{\int_{bottom}^{surface} -g\frac{\partial \eta}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial z^2}\right) dz}$$

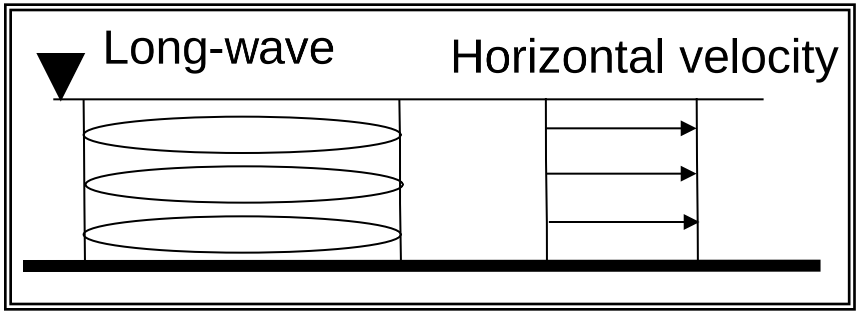
$$= \frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right)$$

1. Differential equation (② Integration of momentum equation)

Integration in vertical direction

(Right side terms)

$$\int_{bottom}^{surface} \boxed{-g\frac{\partial \eta}{\partial x}} + \boxed{\nu\left(\frac{\partial^2 u}{\partial z^2}\right)} dz$$



Horizontal velocity is vertically constant

$$\boxed{\int_{bottom}^{surface} -g\frac{\partial \eta}{\partial x} dz = -gD\frac{\partial \eta}{\partial x}}$$

$$\boxed{\int_{bottom}^{surface} \nu\left(\frac{\partial^2 u}{\partial z^2}\right) dz = \left[\nu\frac{\partial u}{\partial z}\right]_{bottom}^{surface}} = \nu\frac{\partial u_{surface}}{\partial z} - \nu\frac{\partial u_{bottom}}{\partial z} = \frac{1}{\rho}\left(\cancel{\tau_{surface}} - \tau_{bottom}\right)$$

internal shear force of water

Manning formula

$$\boxed{\tau_{bottom} = \frac{\rho g n^2}{D^{7/3}} M|M|}$$

1. Differential equation (② Integration of momentum equation)

(Left side terms)

$$\int_{bottom}^{surface} \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial wu}{\partial z} dz = \frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right)$$

(Right side terms)

$$\int_{bottom}^{surface} -g\frac{\partial \eta}{\partial x} dz = -gD\frac{\partial \eta}{\partial x}$$

$$\int_{bottom}^{surface} \nu\left(\frac{\partial^2 u}{\partial z^2}\right) dz = -\frac{gn^2}{D^{7/3}} M|M|$$

Momentum equation in non-linear long-wave model

$$\boxed{\frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right) + gD\frac{\partial \eta}{\partial x} + \frac{gn^2}{D^{7/3}} M|M| = 0}$$

When I summarize all terms, the momentum equation in non-linear long-wave model is derived.

1. Differential equation (Nonlinear long-wave model)

Continuity equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0$$

Momentum equations (Nonlinear long-wave model)

$$\boxed{\frac{\partial M}{\partial t}} + \boxed{\frac{\partial}{\partial x}\left(\frac{M^2}{D}\right)} + \boxed{gD\frac{\partial \eta}{\partial x}} + \boxed{\frac{gn^2}{D^{7/3}} M|M|} = 0$$

Local change term

Advection term (Nonlinear term)

Gravity term

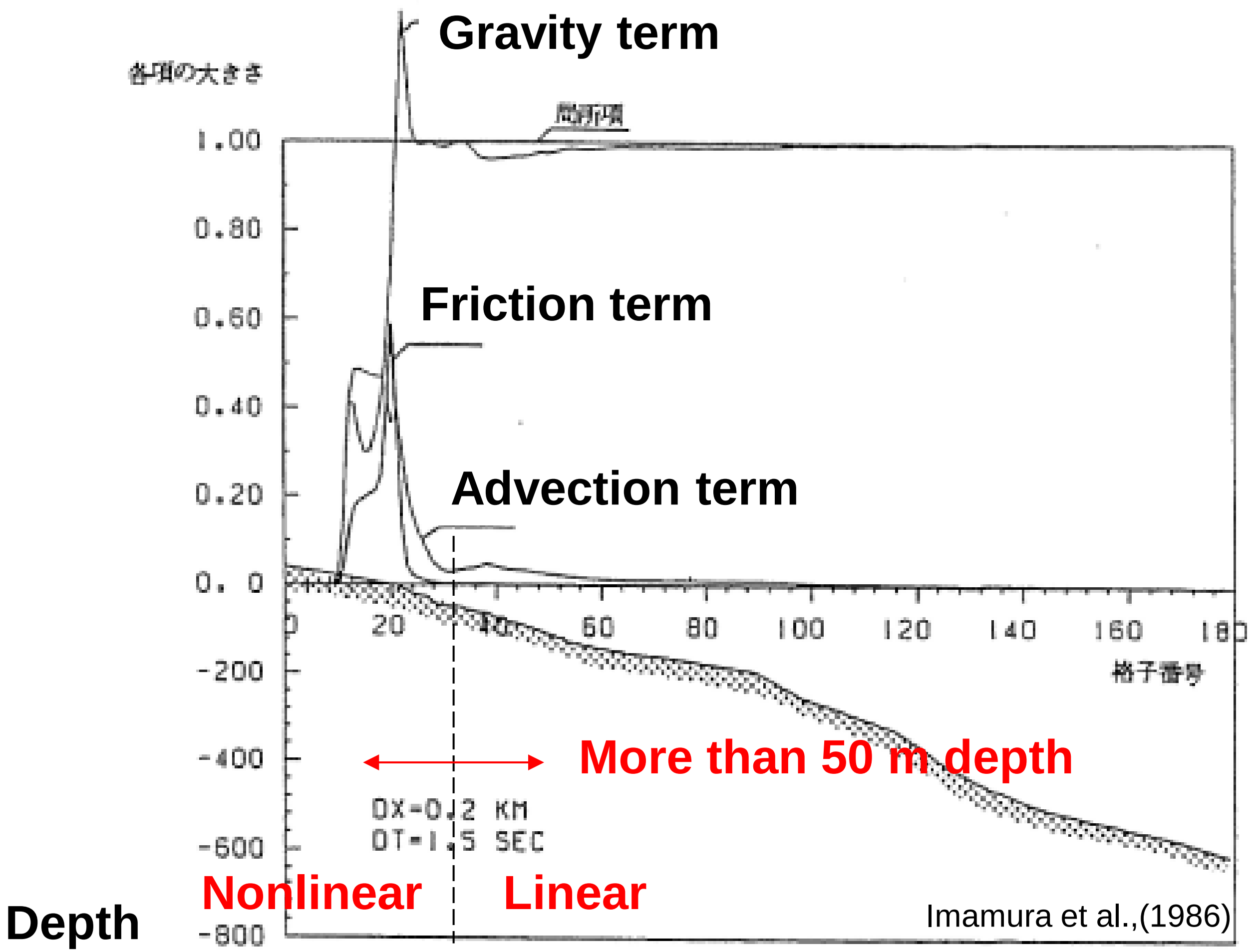
Friction term

Linear long-wave model

$$\frac{\partial M}{\partial t} + gh\frac{\partial \eta}{\partial x} = 0$$

1. Differential equation (Nonlinear long-wave model)

Ratio of each terms to local change term



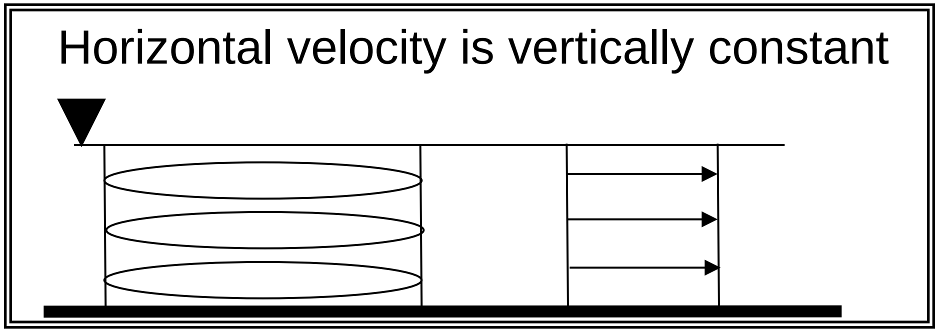
Imamura et al.,(1986)

The limitation of long-wave model

<u>Long wave approximation</u>

1. Long wave (wave height << wave length)
2. Vertical acceleration of water particle ≒ 0

   (Horizontal velocity is vertically constant)
3. The water pressure is gravitational pressure



Long-wave model cannot represent phenomena in which the vertical acceleration is needed such as wave breaking, water splash etc.

Finite difference method

Outline of tsunami modeling

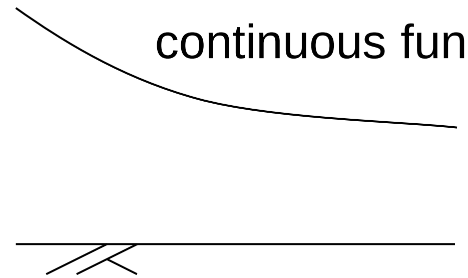
1. Derivation of governing equations

Differential equation

$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0$$

$$\frac{\partial M}{\partial t} + gh\frac{\partial \eta}{\partial x} = 0$$

continuous function



2. Discretization of governing equations

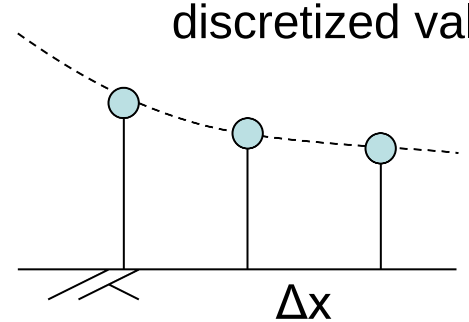
The differential equations could not be computed directly.

Finite difference method

$$\frac{\eta^{k+1} - \eta^{k}}{\Delta t} + \frac{M_{i+1/2} - M_{i-1/2}}{\Delta x} = 0$$

$$\frac{M^{k+1/2} - M^{k-1/2}}{\Delta t} + gh_i \frac{\eta_{i+1} - \eta_i}{\Delta x} = 0$$

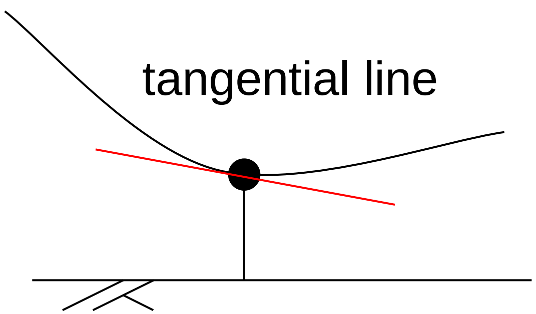
discretized value



3. Programming using fortran

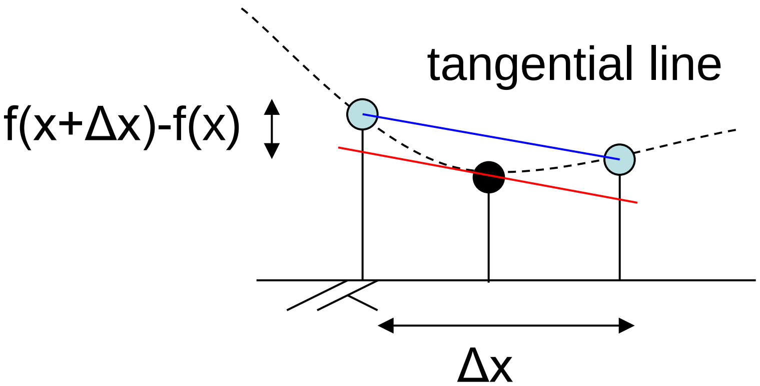
Basic concept

1. Differential equation



The differential of a function means tangential line.

2. Finite difference equation



The differential is approximated by the tangential line from discretized values

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The finite difference equation

Forward difference

$$\frac{\partial f(x)}{\partial x} \approx \frac{f_{i+1} - f_i}{\Delta x}$$

low accuracy, stable (M>0)

Backward difference

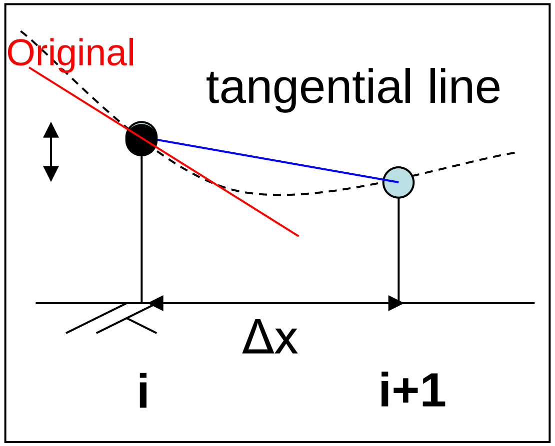
$$\frac{\partial f(x)}{\partial x} \approx \frac{f_i - f_{i-1}}{\Delta x}$$

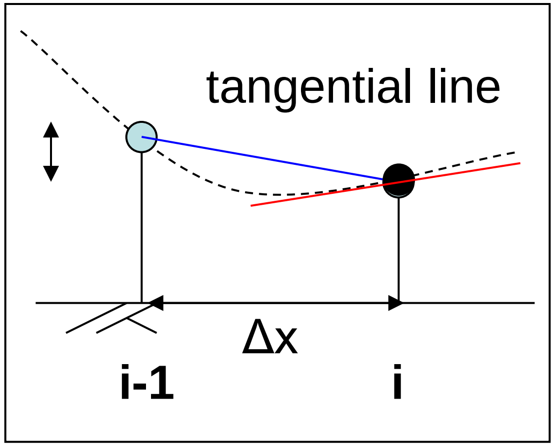
Low accuracy, stable (M<0)

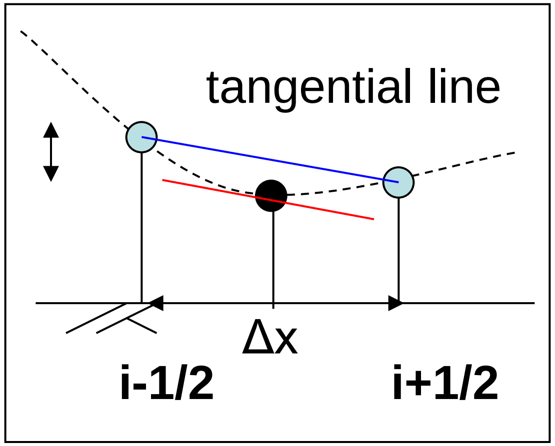
Centered difference

$$\frac{\partial f(x)}{\partial x} \approx \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x}$$

high accuracy, Including unstable error







Practice (1)

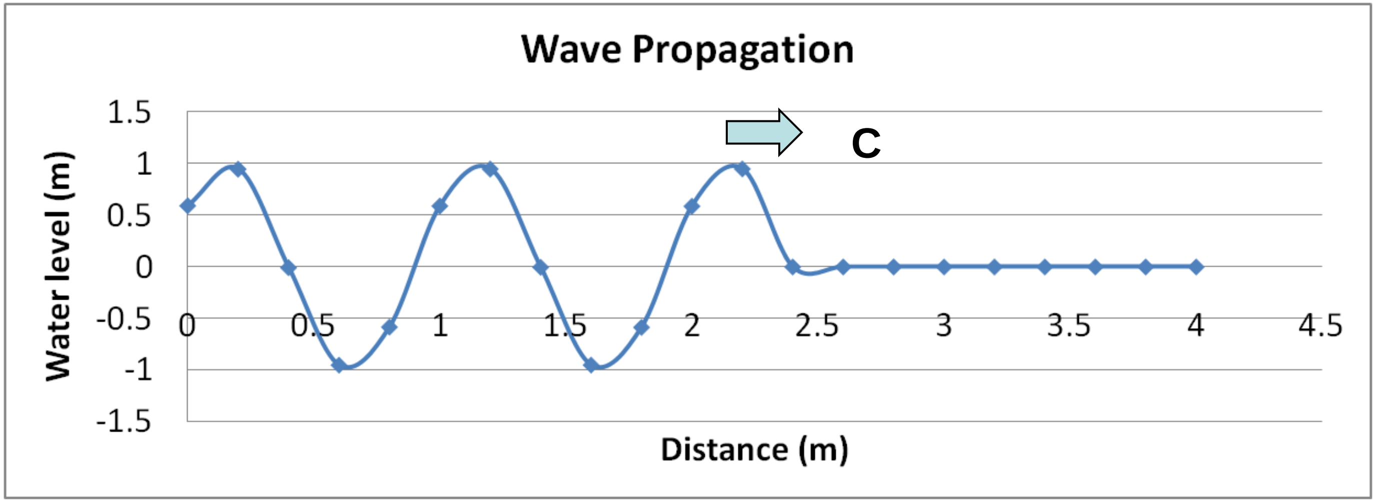
Discretize the following equation using (1) backward and (2) centered difference for spatial grid and backward difference for time grid

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}$$

t: time

x: distance

C: wave speed



Practice (1)

Discretize the following equation using backward and centered difference for spatial grid and backward difference for time grid

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}$$

t: time

x: distance

C: wave speed

Backward difference

$$\frac{u_i^k - u_i^{k-1}}{\Delta t} = c\frac{u_i^k - u_{i-1}^k}{\Delta x}$$

Centered difference

$$\frac{u_i^k - u_i^{k-1}}{\Delta t} = c\frac{u_{i+1}^k - u_{i-1}^k}{\Delta x}$$

Practice (2)

Simulate the equation using Excel in the following conditions

① dx=0.2 (m)   dt=0.2 (sec)   c=1.0 (m/s)

② Area = 0～ 10 (m), Total time = 5 (sec)

③ Boundary(sin wave with 1 m amplitude and 1.0 sec period) => sin(2*3.14/1.0)

Practice (2)

(2) Simulate the equation using Excel using the following conditions

① dx=0.2 (m)   dt=0.2 (sec)   c=1.0 (m/s)

② Area = 0～ 10 (m), Total time = 5 (sec)

③ Boundary(sin wave with 1 m amplitude and 1.0 sec period) => sin(2*3.14/1.0)

Step 1 make the following table

Distance \ time		0	1	2	…
		0	0.2	0.4	… 5
0	0				
1	0.2				

Practice (2)

(2) Simulate the equation using Excel using the following conditions

① dx=0.2 (m)   dt=0.2 (sec)   c=1.0 (m/s)

② Area = 0～ 10 (m), Total time = 5 (sec)

③ Boundary(sin wave with 1 m amplitude and 1.0 sec period) => sin(2*3.14/1.0)

Step 2 Input the initial conditions at time 0

Distance \ time		0	1	2	…
		0	0.2	0.4	… 5
0	0	0			
1	0.2	0			

Practice (2)

(2) Simulate the equation using Excel using the following conditions

① dx=0.2 (m)   dt=0.2 (sec)   c=1.0 (m/s)

② Area = 0～ 10 (m), Total time = 5 (sec)

③ Boundary(sin wave with 1 m amplitude and 1.0 sec period) => sin(2*3.14/1.0)

Step 3 Input the boundary condition

Distance \ time		0	1	2	…
		0	0.2	0.4	… 5
0	0	0	=sin(2*3.14/1.0*time0)		
1	0.2	0			

Practice (2)

(2) Simulate the equation using Excel using the following conditions

① dx=0.2 (m)   dt=0.2 (sec)   c=1.0 (m/s)

② Area = 0～ 10 (m), Total time = 5 (sec)

③ Boundary(sin wave with 1 m amplitude and 1.0 sec period) => sin(2*3.14/1.0)

<u>Step 4 Input the difference equation</u>

Distance \ time		0	1	2	…
		0	0.2	0.4	… 5
0	0	0	=sin(2*3.14/1.0 *time1)		
1	0.2	0	u11=u01-c*dt/dx*(u01-u00)		

Practice (2)

(2) Simulate the equation using Excel using the following conditions

① dx=0.2 (m)   dt=0.2 (sec)   c=1.0 (m/s)

② Area = 0～ 10 (m), Total time = 5 (sec)

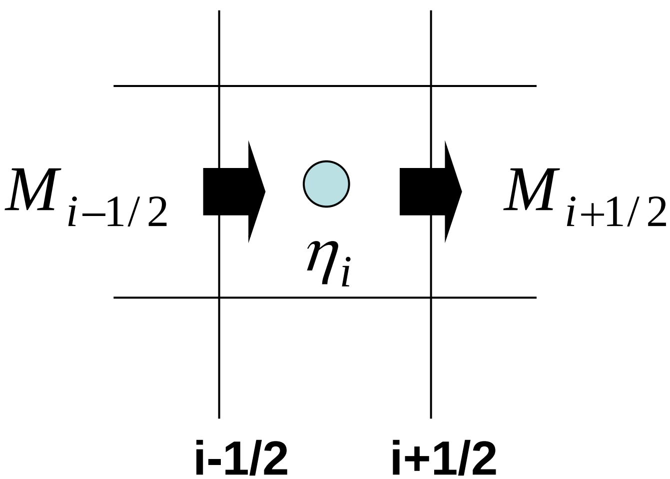
③ Boundary(sin wave with 1 m amplitude and 1.0 sec period) => sin(2*3.14/1.0)

Step 4 Input the difference equation

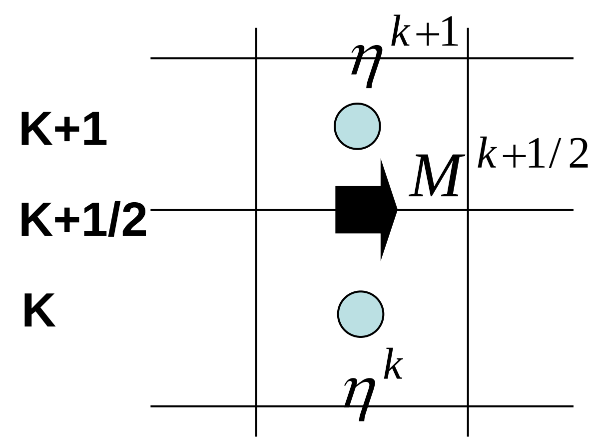
time / Distance		0	1	2	…
		0	0.2	0.4	… 5
0	0	0	=sin(2*3.14/1.0 *time1)		
1	0.2	0	u11=u01-c*dt/dx*(u01-u00)		

Staggered grid

(Special grid)



(Time grid)



<u>**Water elevation is defined at center of grid**</u>

<u>**Water flux is defined at edge of grid**</u>

there is dislocation between η and M

The finite difference equation (Linear term)

Continuity equation

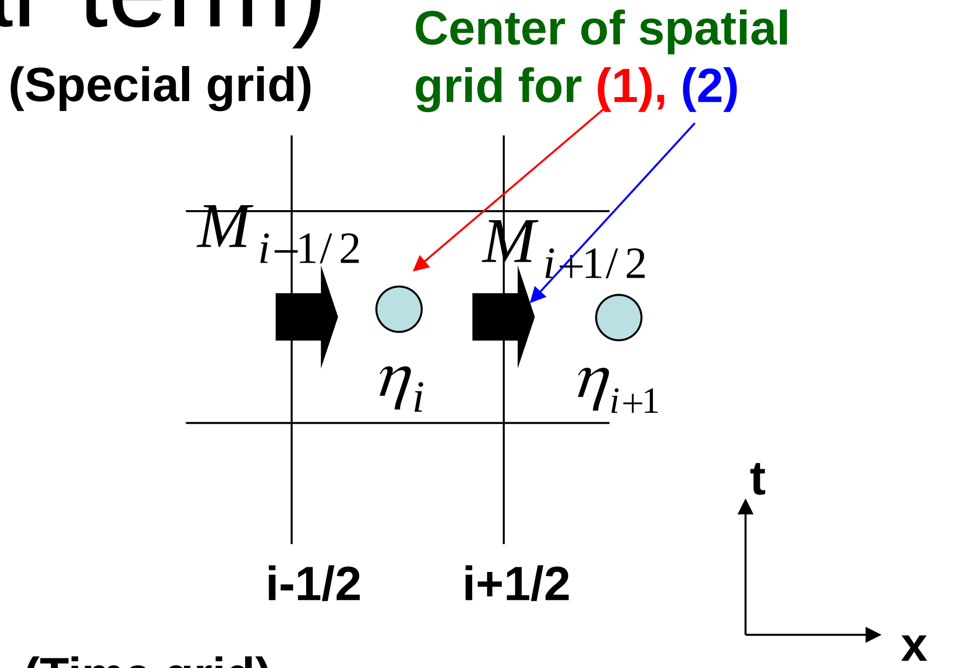
$$\frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0 \qquad \textbf{(1)}$$

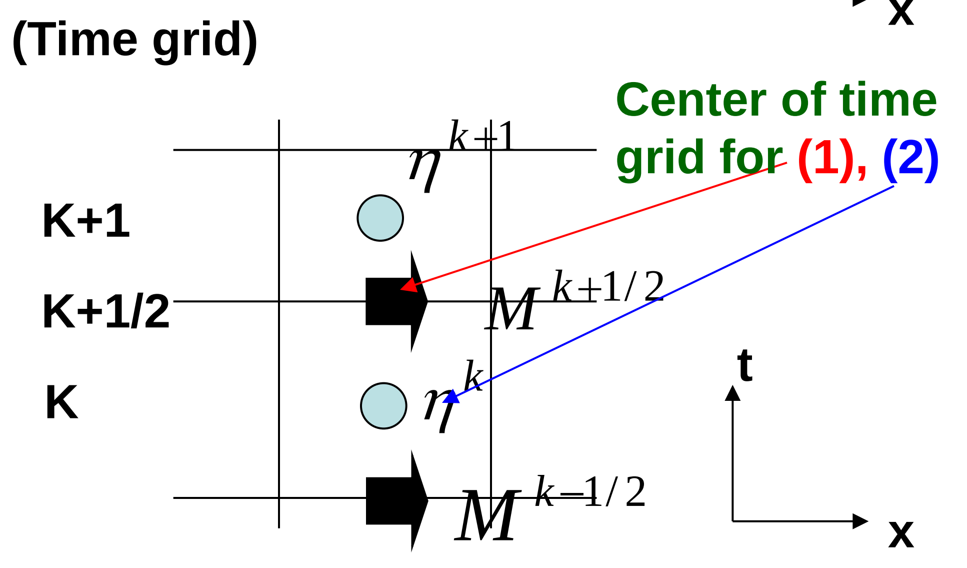
$$\Rightarrow \quad \frac{\eta^{k+1} - \eta^{k}}{\Delta t} + \frac{M_{i+1/2} - M_{i-1/2}}{\Delta x} = 0$$

Momentum equation

$$\frac{\partial M}{\partial t} + gh\frac{\partial \eta}{\partial x} = 0 \qquad \textbf{(2)}$$

$$\Rightarrow \quad \frac{M^{k+1/2} - M^{k-1/2}}{\Delta t} + gD_{i+1/2}\frac{\eta_{i+1} - \eta_{i}}{\Delta x} = 0$$





The finite difference equation

Continuity equation

$$\frac{\boxed{\eta^{k+1}} - \eta^{k}}{\Delta t} + \frac{M_{i+1/2} - M_{i-1/2}}{\Delta x} = 0 \quad \Rightarrow \quad \boxed{\eta^{k+1}} = \boxed{\eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)}$$

Next time step Previous time step

Momentum equation

$$\frac{\boxed{M^{k+1/2}} - M^{k-1/2}}{\Delta t} + gD_{i+1/2}\frac{\eta_{i+1} - \eta_{i}}{\Delta x} = 0 \quad \Rightarrow \quad \boxed{M^{k+1/2}} = \boxed{M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_{i}\right)}$$

Next time step Previous time step

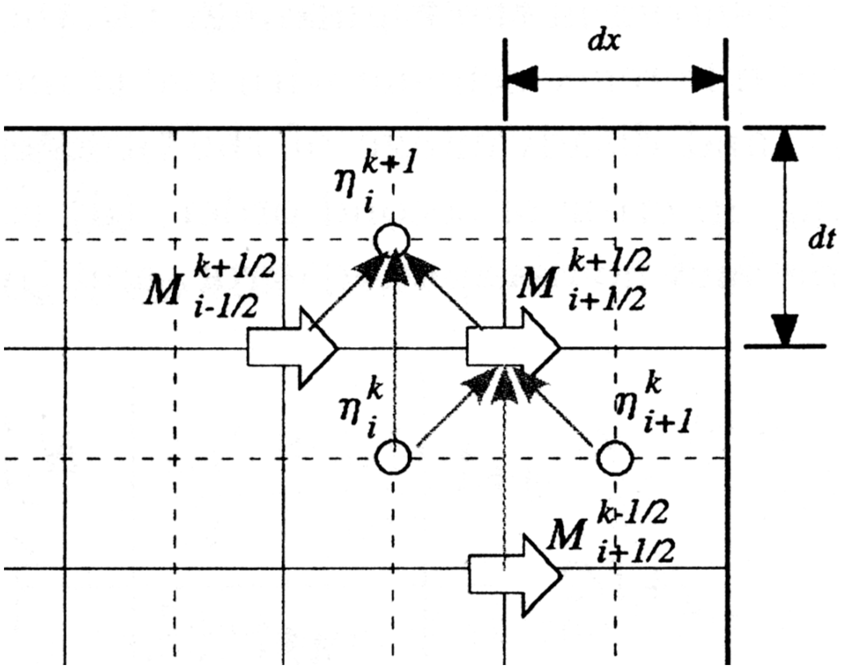
The finite difference equation

Continuity equation

$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(\boxed{M_{i+1/2} - M_{i-1/2}}\right)$$

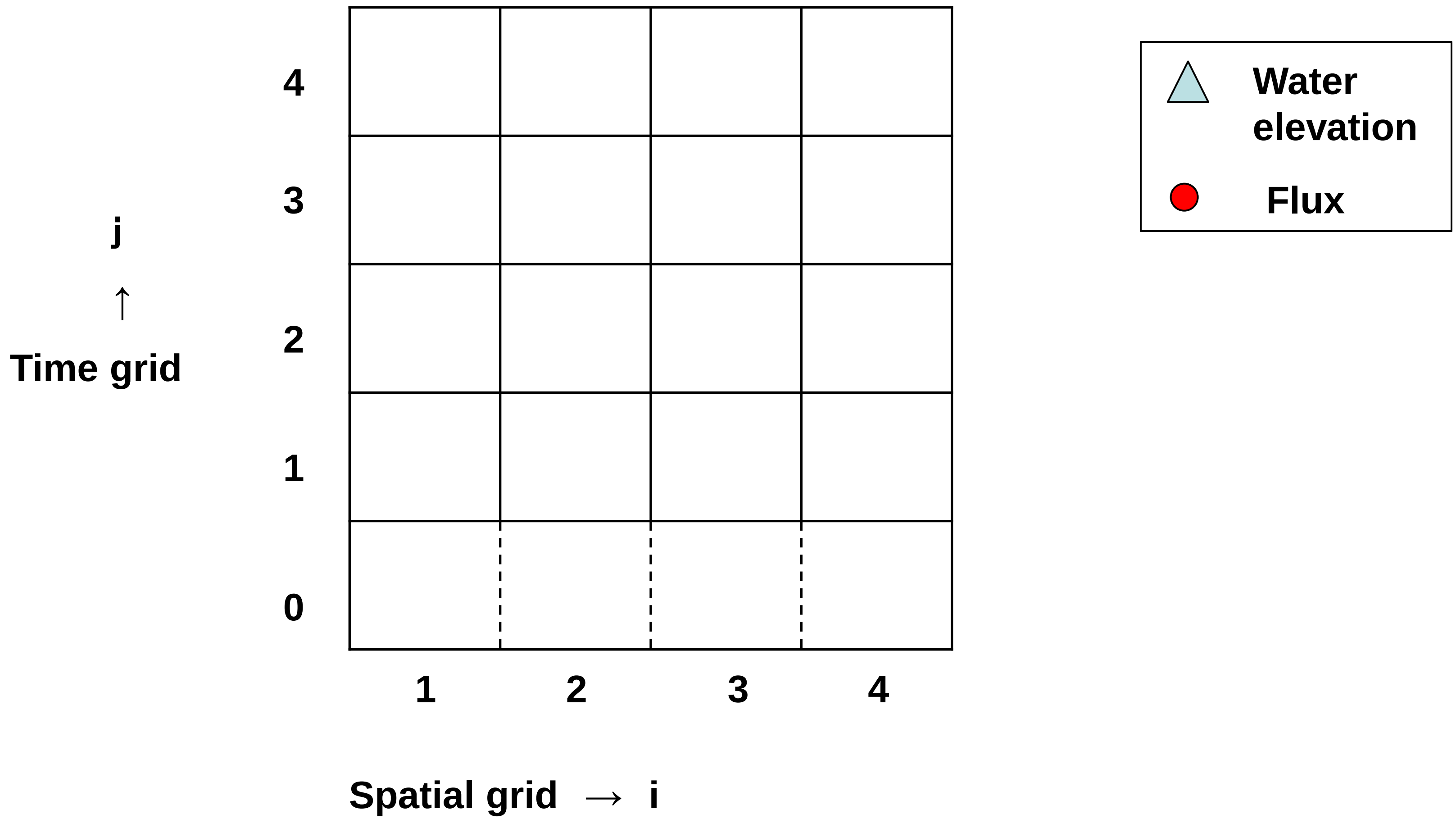
Momentum equation

$$M^{k+1/2} = M^{k-1/2} - gh_{i+1/2}\,\boxed{\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_{i}\right.}\left.\right)$$



Practice (2)

- Write the leap-frog method on the flowing grid



Practice (2)

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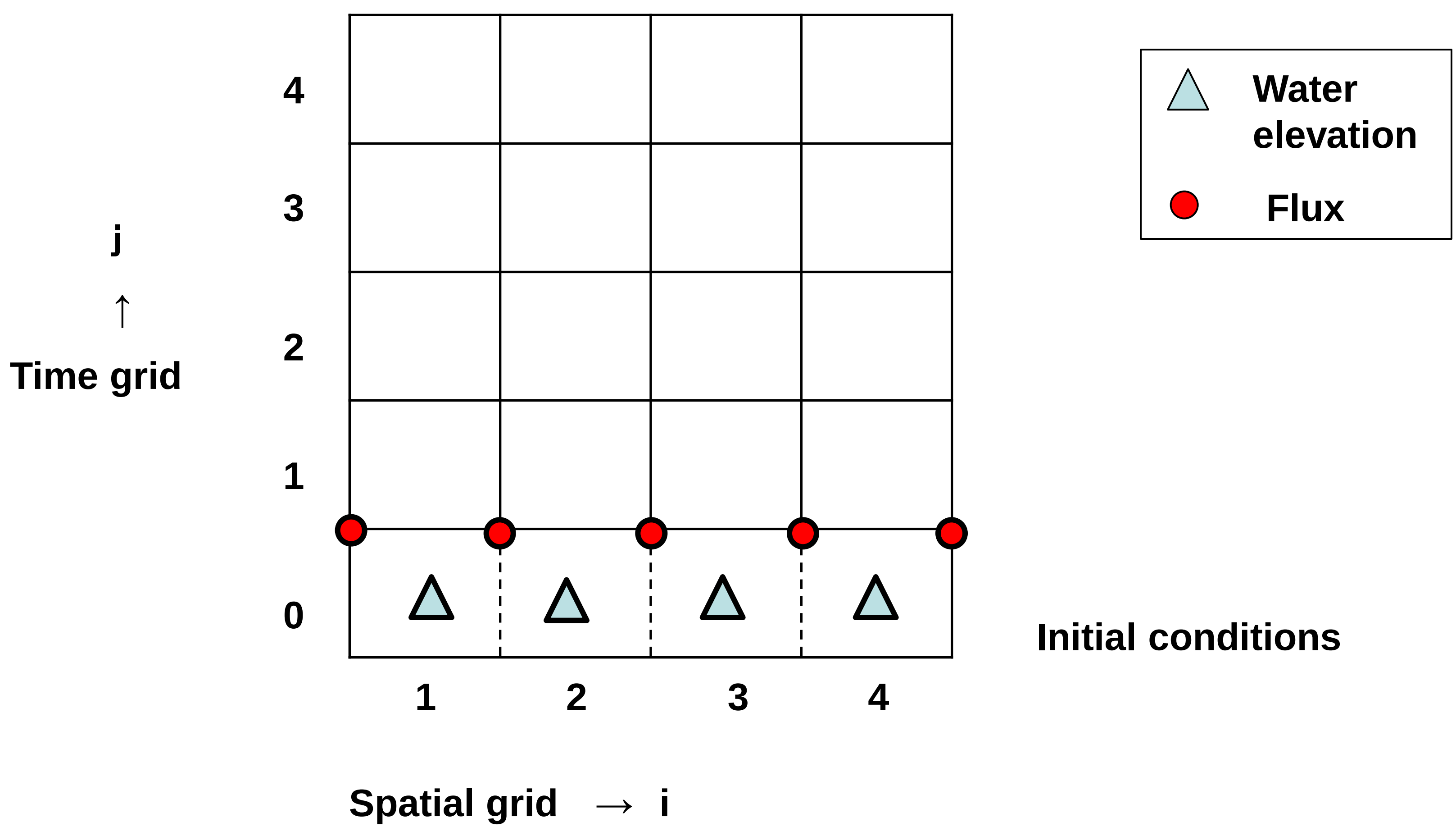

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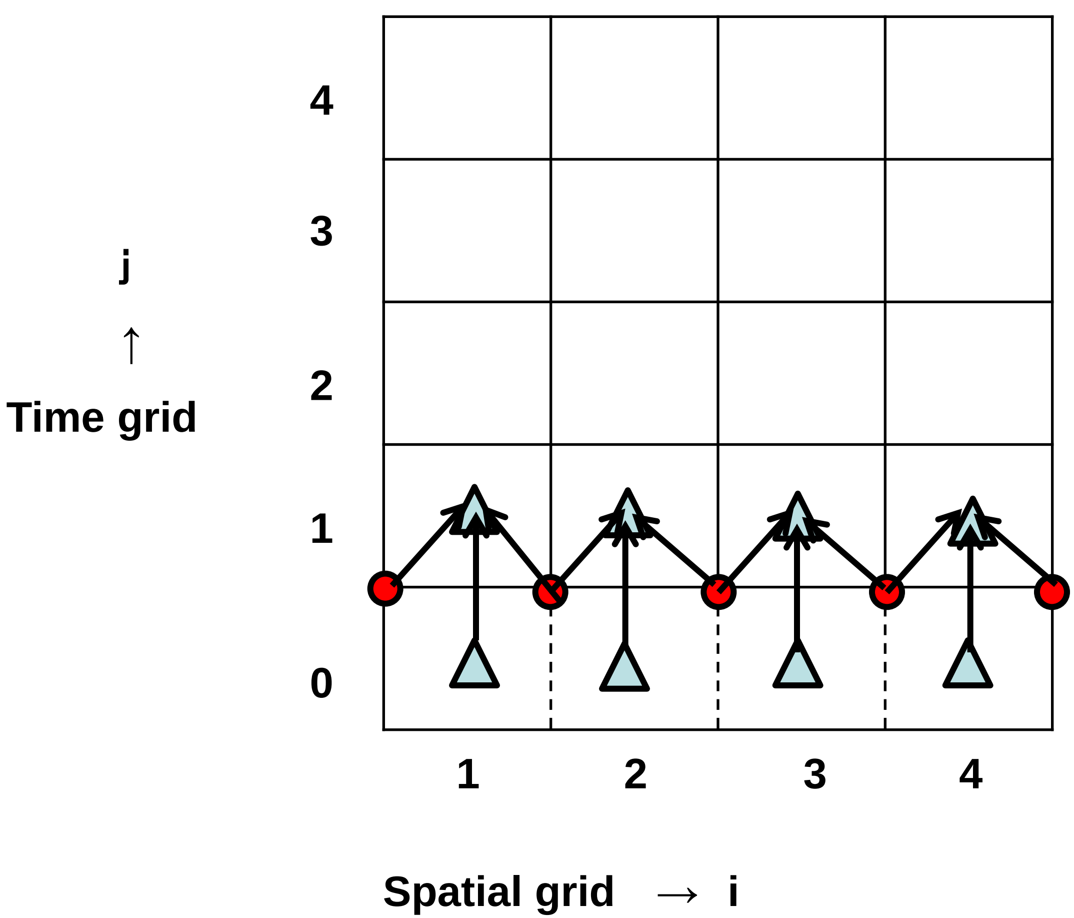
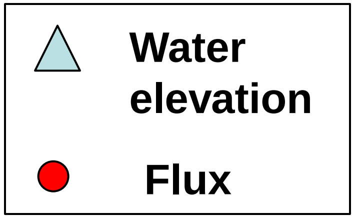
- Write the leap-frog method on the flowing grid





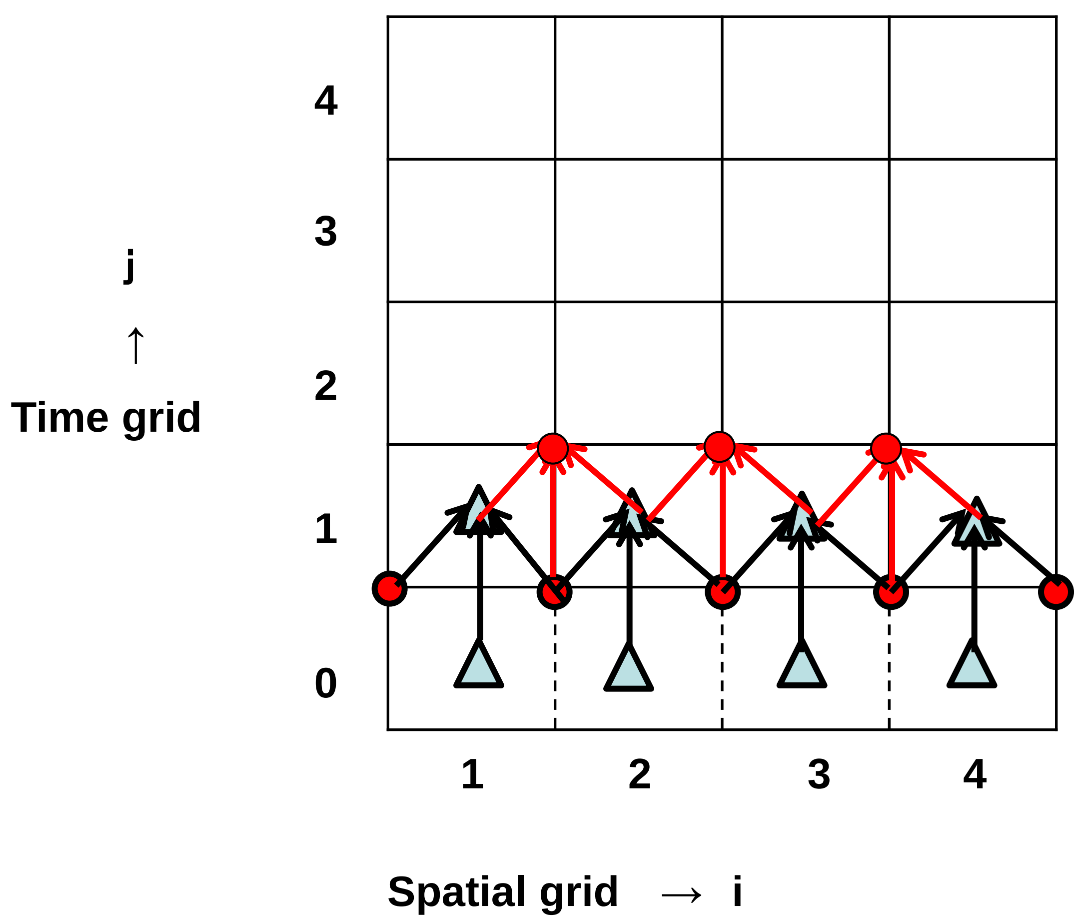
Continuity equation

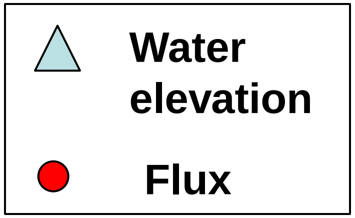
$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

Practice (2)

- Write the leap-frog method on the flowing grid





Moment equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_i\right)$$

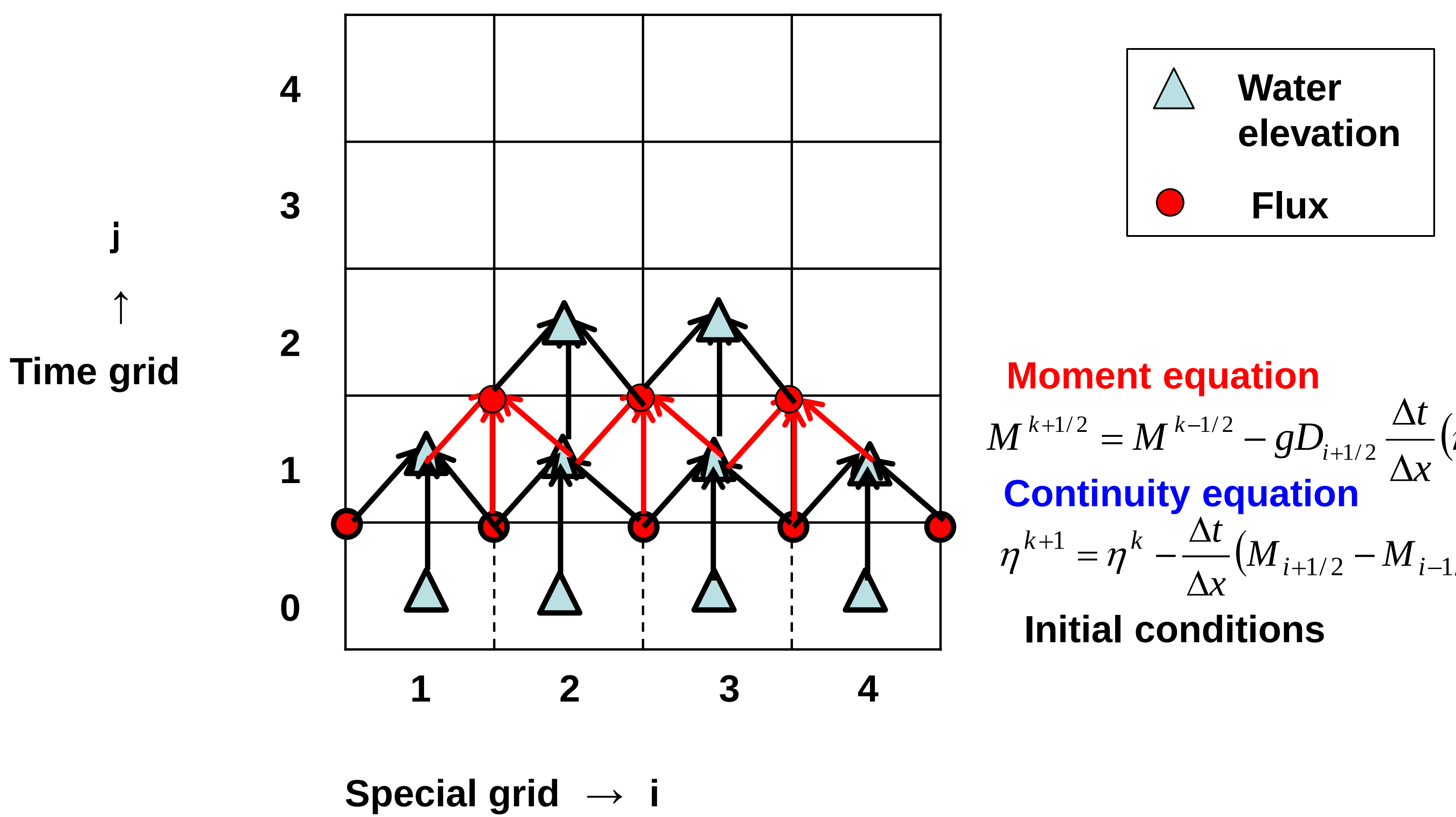
Continuity equation

$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

Practice (2)

- Write the leap-frog method on the flowing grid



Moment equation

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Initial conditions

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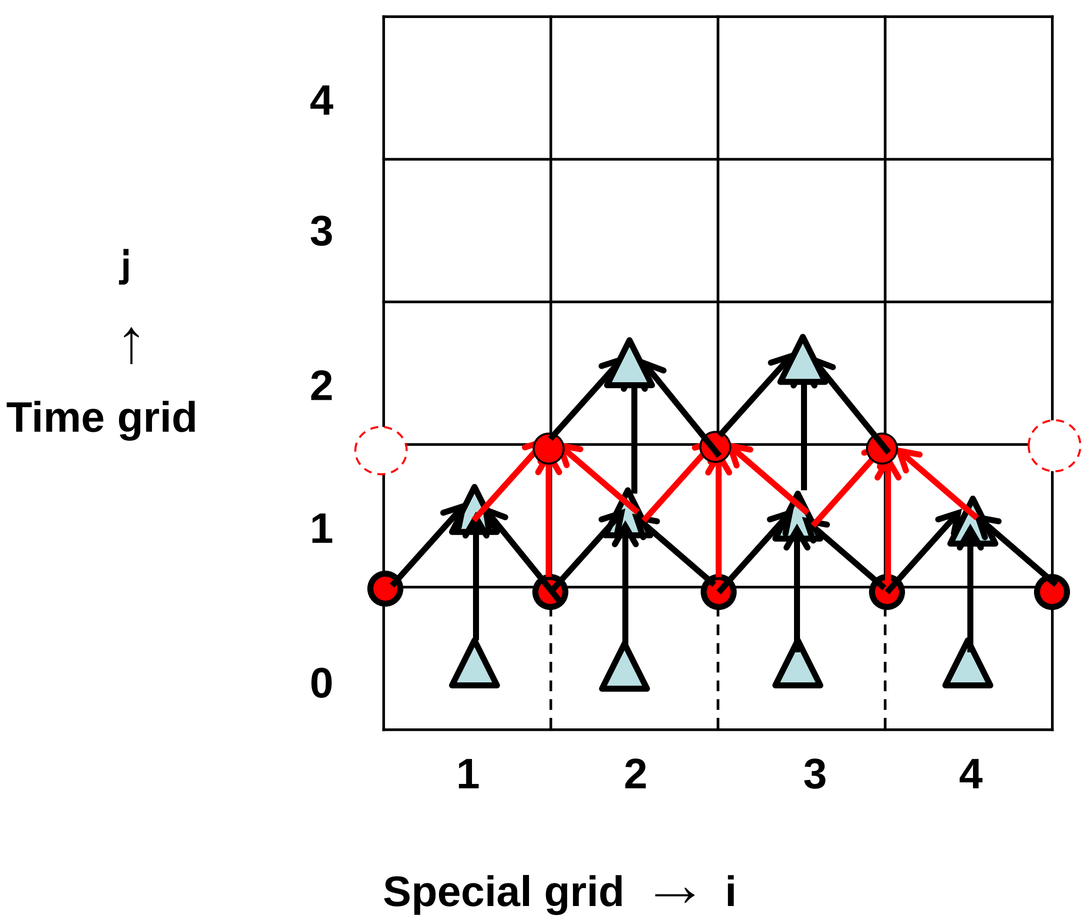
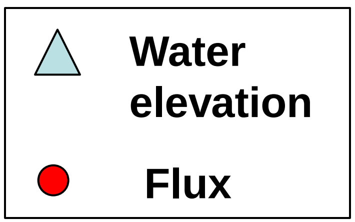




Moment equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_i\right)$$

Continuity equation

$$\eta^{k+1} = \eta^k - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

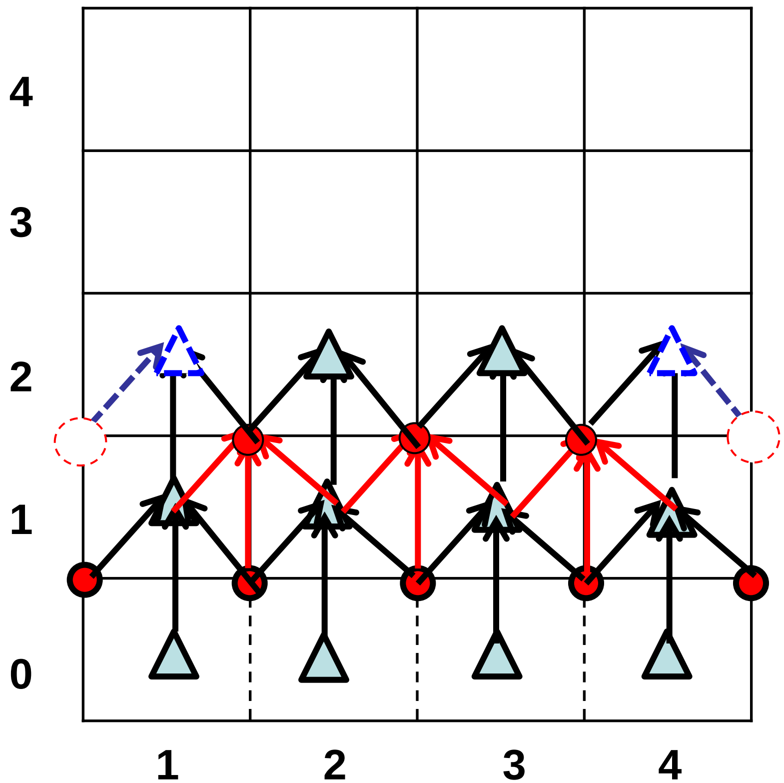
Practice (2)

- Write the leap-frog method on the flowing grid

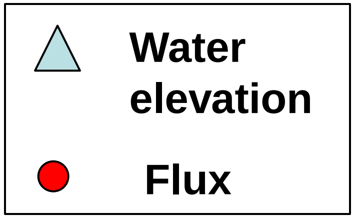
j

↑

Time grid



Special grid → i



Moment equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_i\right)$$

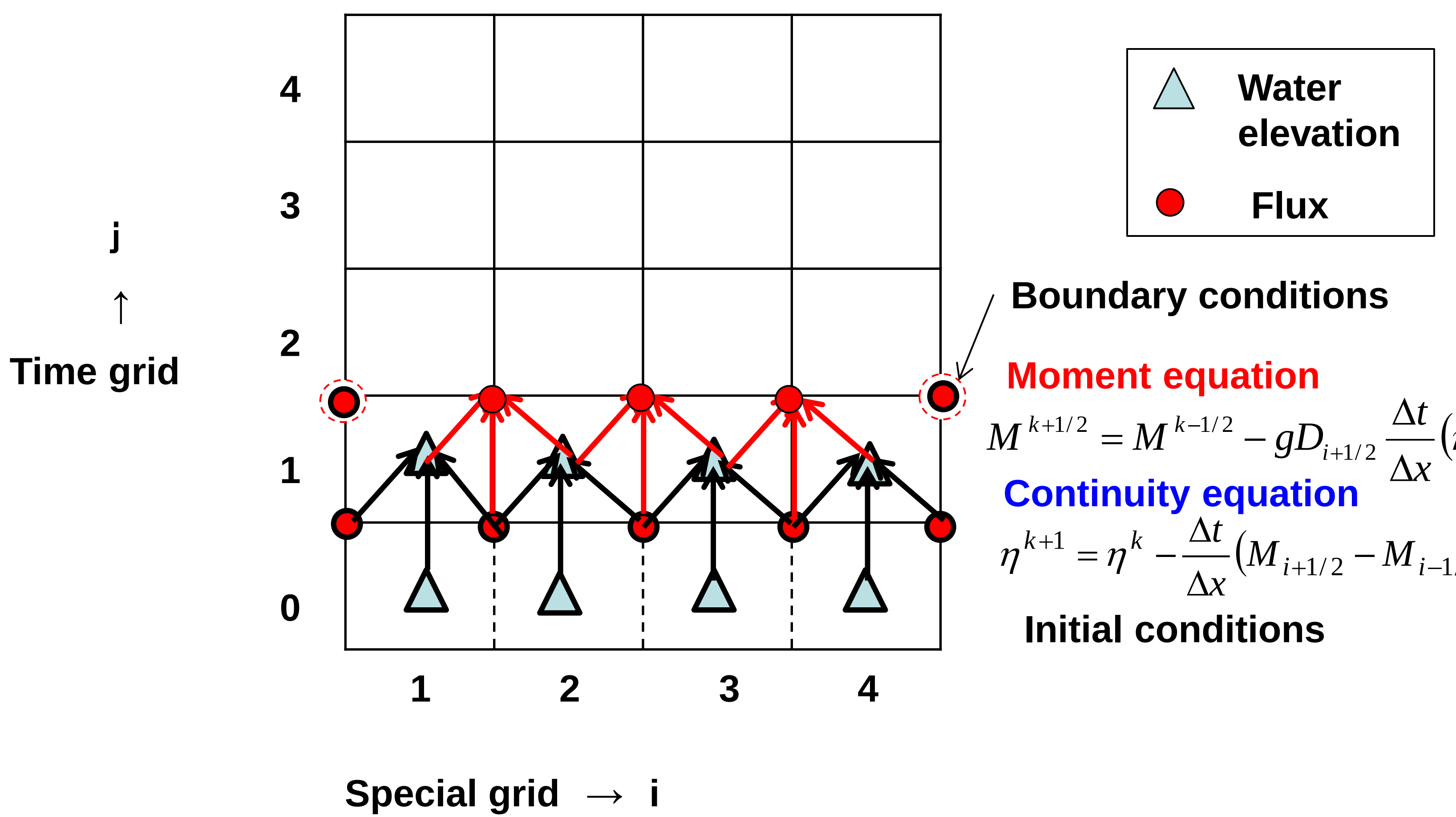
Continuity equation

$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

Practice (2)

- Write the leap-frog method on the flowing grid



Boundary conditions

Moment equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_i\right)$$

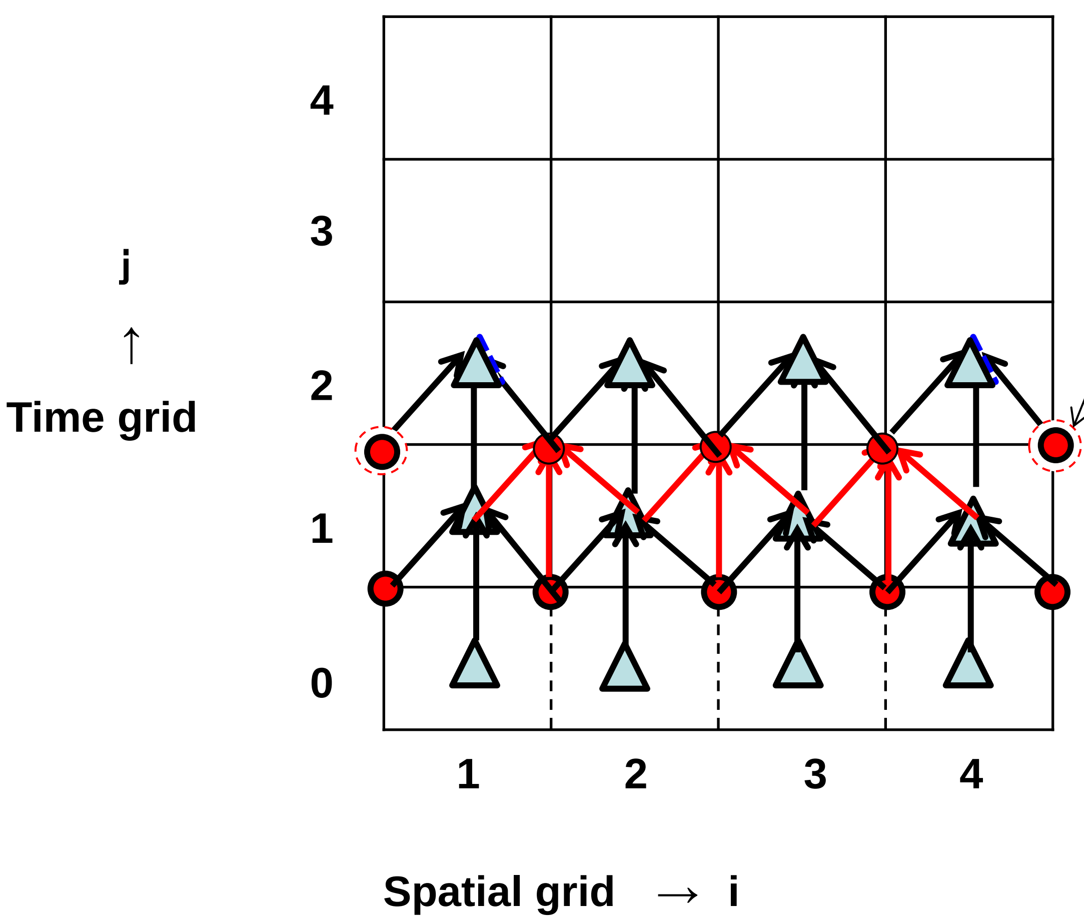
Continuity equation

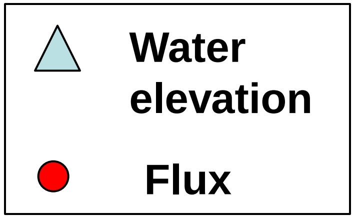
$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

Practice (2)

- Write the leap-frog method on the flowing grid





Boundary conditions

Moment equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_i\right)$$

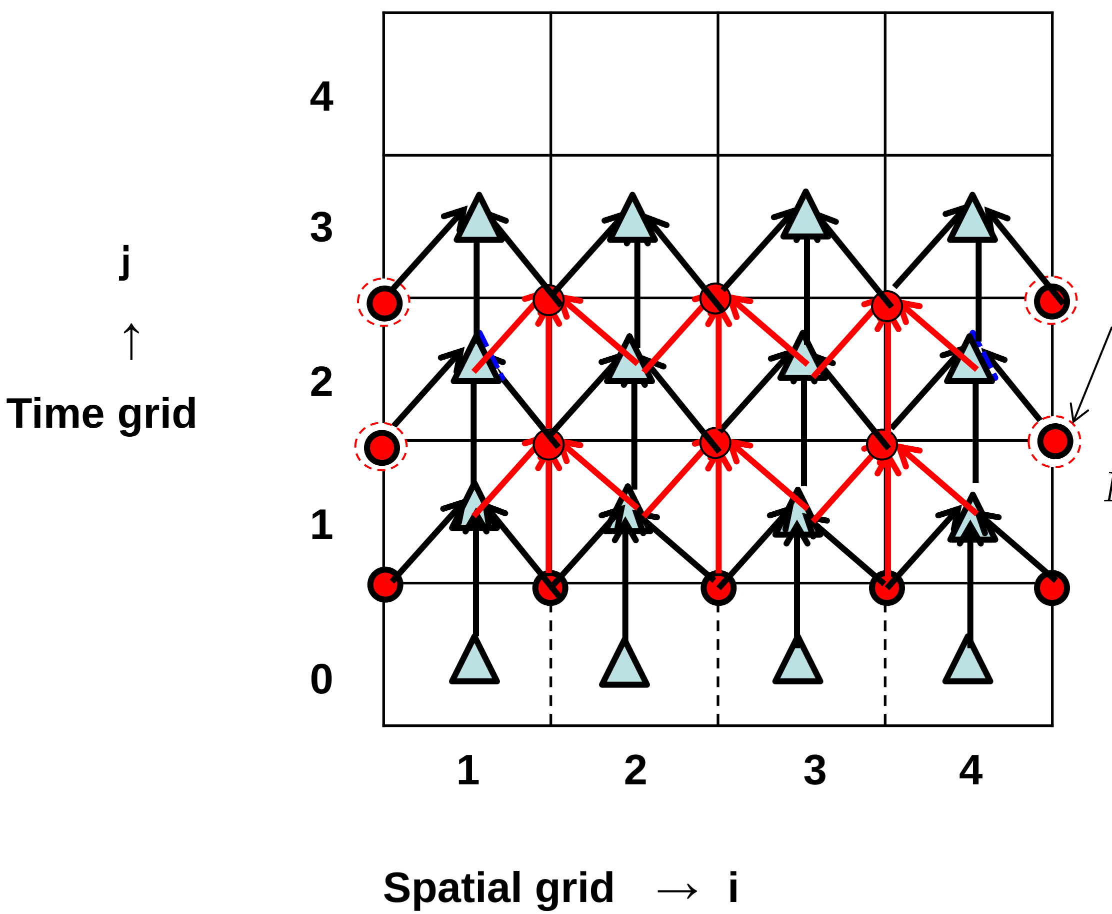
Continuity equation

$$\eta^{k+1} = \eta^k - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

Practice (2)

- Write the leap-frog method on the flowing grid



Water elevation

Flux

Boundary conditions

Moment equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_i\right)$$

Continuity equation

$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

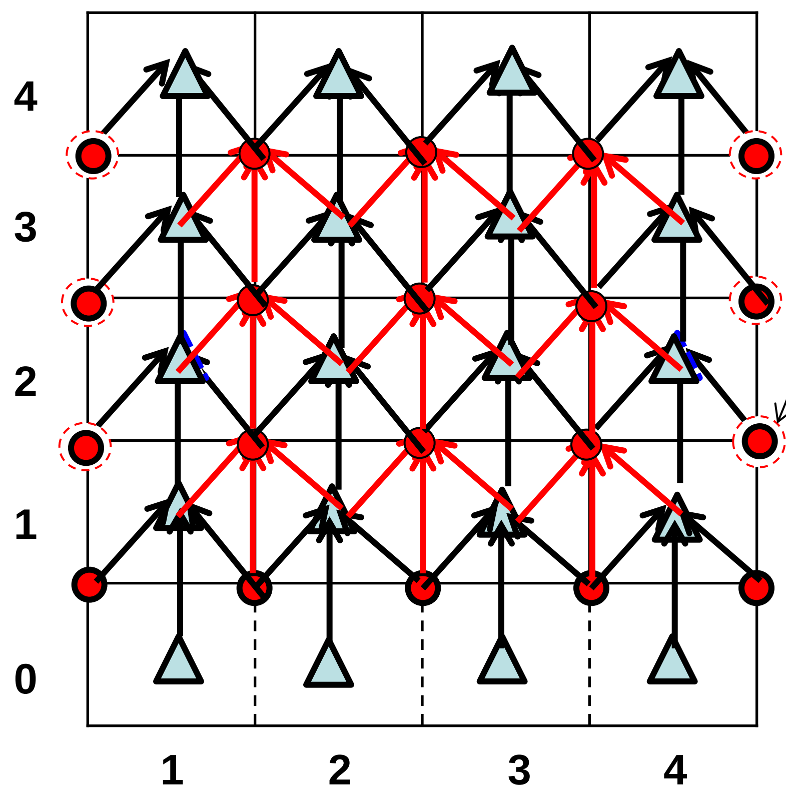
Practice (2)

- Write the leap-frog method on the flowing grid

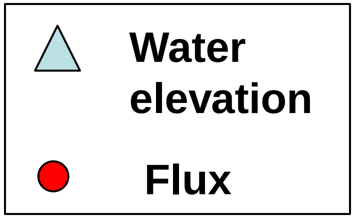
j

↑

Time grid



Spatial grid $\rightarrow$ i



Boundary conditions

Moment equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\Delta t}{\Delta x}\left(\eta_{i+1} - \eta_i\right)$$

Continuity equation

$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Initial conditions

The finite difference equation (Nonlinear term)

Nonlinear terms cause unstable effect in some cases.

Thus, more stable scheme is used for nonlinear terms.

Upwind difference scheme

$$\frac{\partial}{\partial x}\left(\frac{M^2}{D}\right)$$

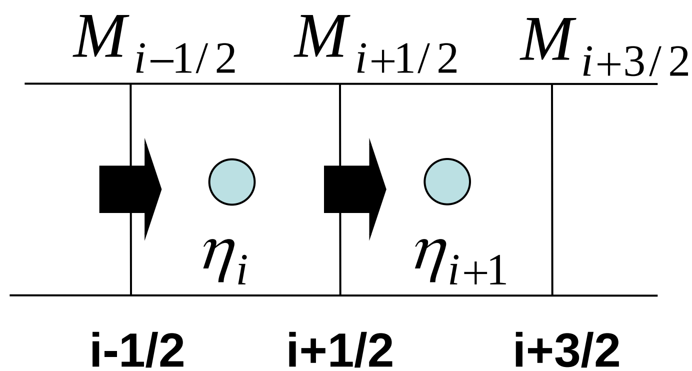
Backward difference (M > 0)

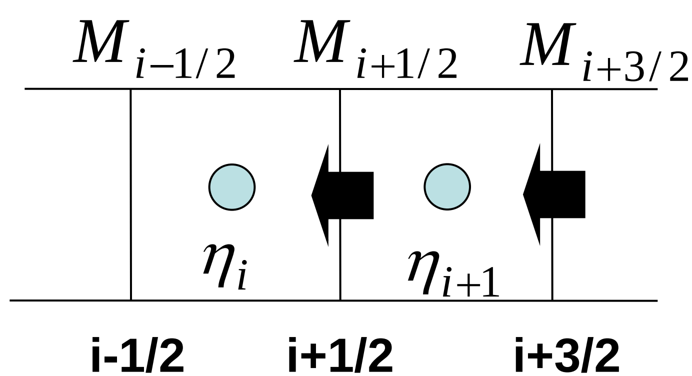
$$\frac{\partial}{\partial x}\left(\frac{M^2}{D}\right) \approx \frac{1}{\Delta x}\left(\frac{M^2_{i+1/2}}{D_{i+1/2}} - \frac{M^2_{i-1/2}}{D_{i-1/2}}\right)$$

Forward difference (M < 0)

$$\frac{\partial}{\partial x}\left(\frac{M^2}{D}\right) \approx \frac{1}{\Delta x}\left(\frac{M^2_{i+3/2}}{D_{i+3/2}} - \frac{M^2_{i+1/2}}{D_{i+1/2}}\right)$$

Special grid





The finite difference equation of tsunami model

Continuity equation

$$\eta^{k+1} = \eta^{k} - \frac{\Delta t}{\Delta x}\left(M_{i+1/2} - M_{i-1/2}\right)$$

Momentum equation

$$M^{k+1/2} = M^{k-1/2} - gD_i \frac{\eta_{i+1} - \eta_i}{\Delta x}\Delta t - \left(\lambda_1 \frac{M_{i+3/2}^2}{D_{i+3/2}} + \lambda_2 \frac{M_{i+1/2}^2}{D_{i+1/2}} + \lambda_3 \frac{M_{i-1/2}^2}{D_{i-1/2}}\right)$$

$$\begin{cases} M_{i+1/2} \geq 0 & \lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = -1 \\ M_{i+1/2} < 0 & \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0 \end{cases}$$

Inundation model

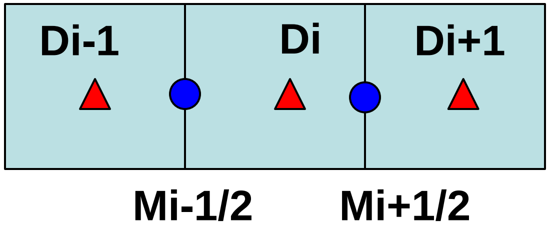
Inundation model

Momentum equation

$$M^{k+1/2} = M^{k-1/2} - gD_{i+1/2}\frac{\eta_{i+1}-\eta_i}{\Delta x}\Delta t - \left(\lambda_1 \frac{M^2_{i+3/2}}{D_{i+3/2}} + \lambda_2 \frac{M^2_{i+1/2}}{D_{i+1/2}} + \lambda_3 \frac{M^2_{i-1/2}}{D_{i-1/2}}\right)$$

D (total depth) in momentum equation is defined at the edge of grid,

But estimated D(total depth) from continuity equation is center of grid



Di-1/2=0.5*(Di + Di-1)

Di+1/2=0.5*(Di+1 + Di)

IF Di = 0 (dry condition)

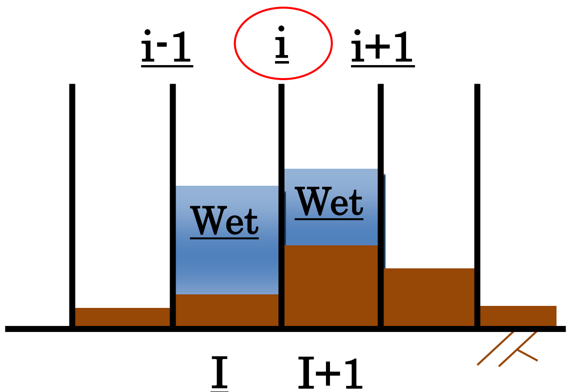
･･･How to estimate Di+1/2

⇒ Inundation model

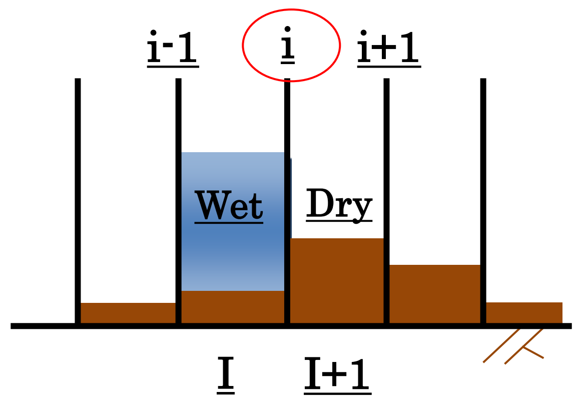
Inundation model

■ **There are 6 type of inundation patterns in difference scheme**

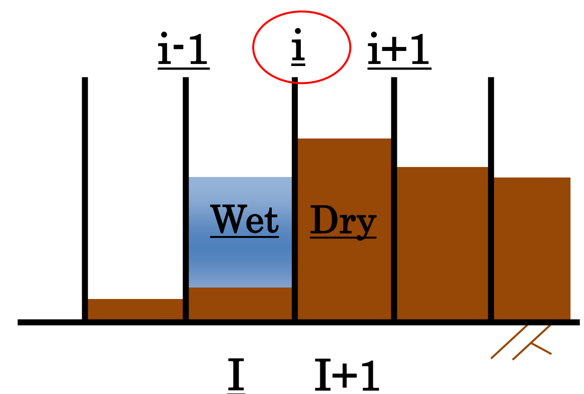
Pattern (1)



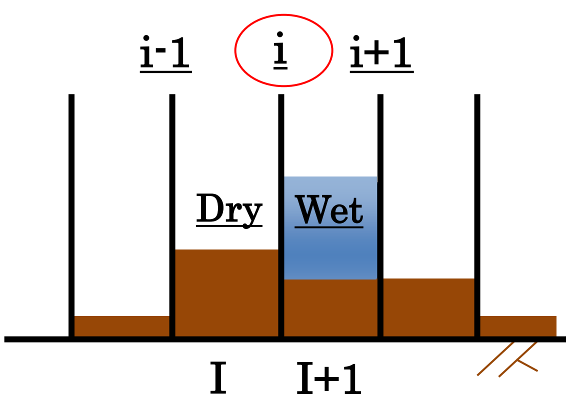
Pattern (2)



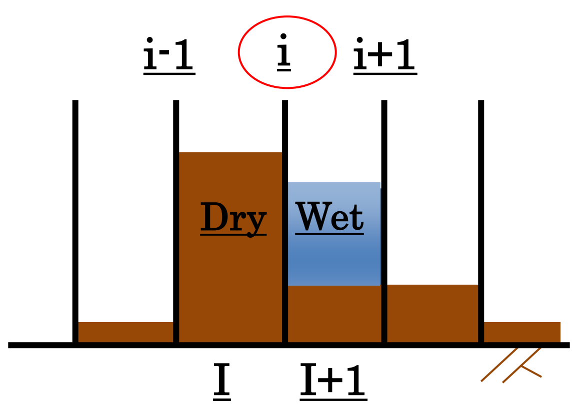
Pattern (3)



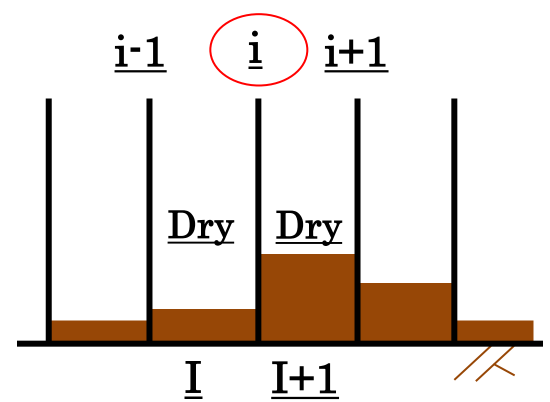
Pattern (4)



Pattern (5)

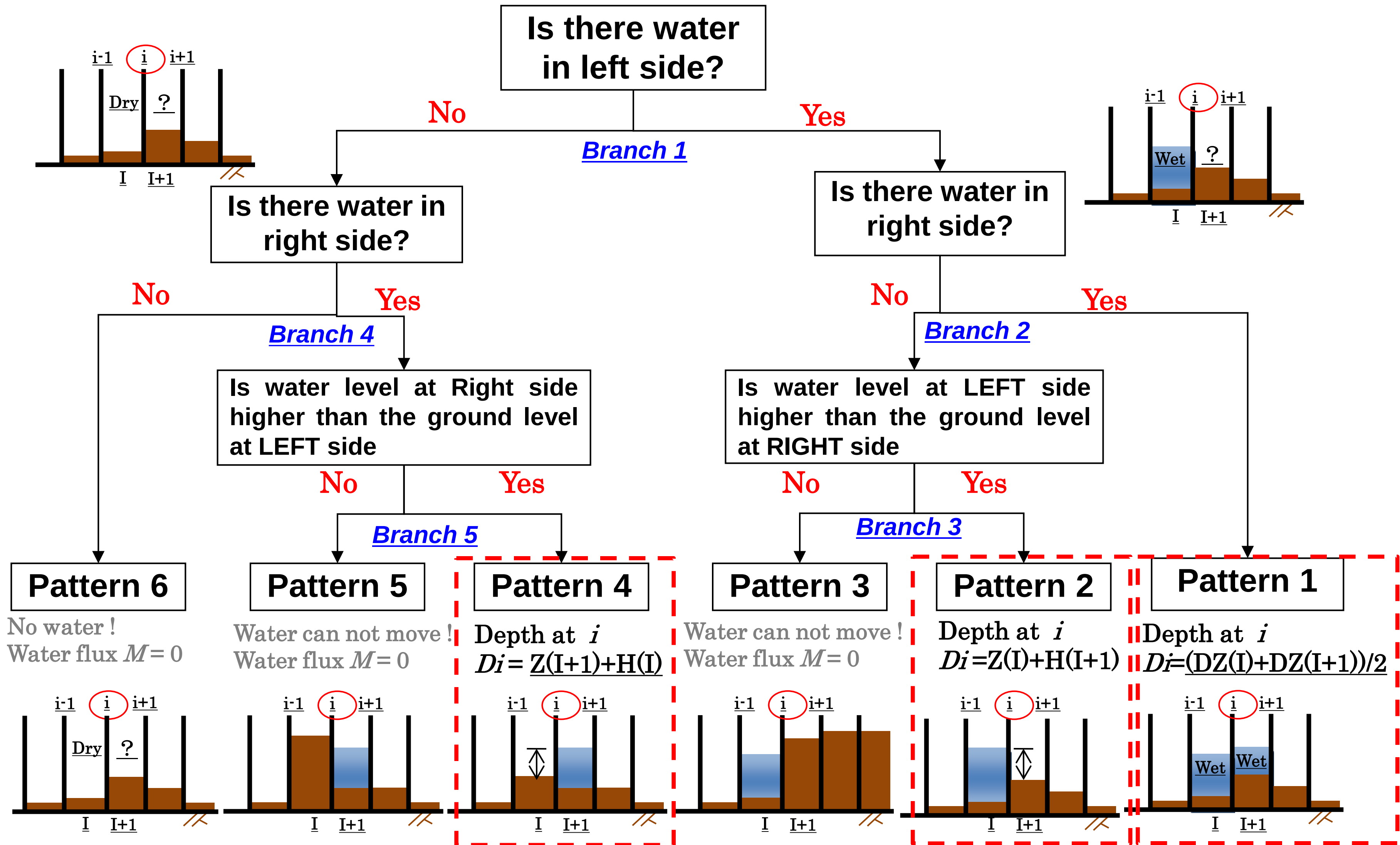


Pattern (6)



Inundation model

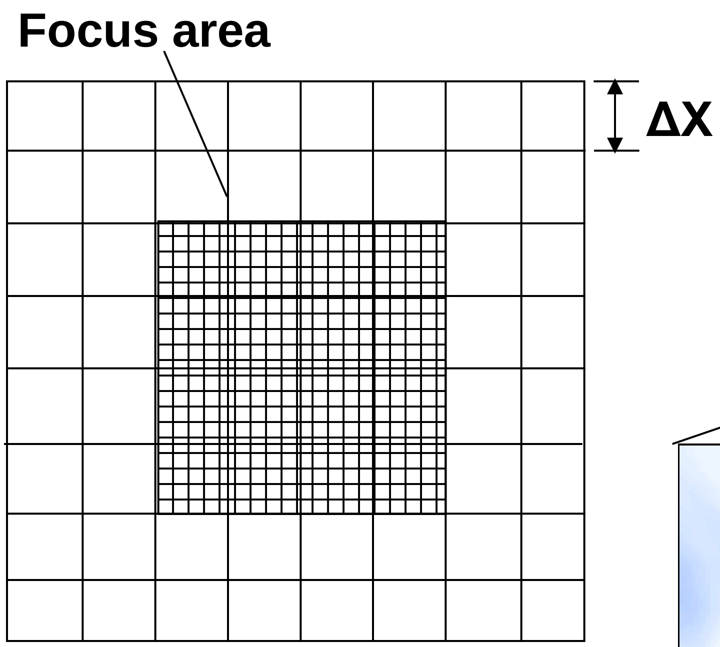
■ Algorism of inundation model

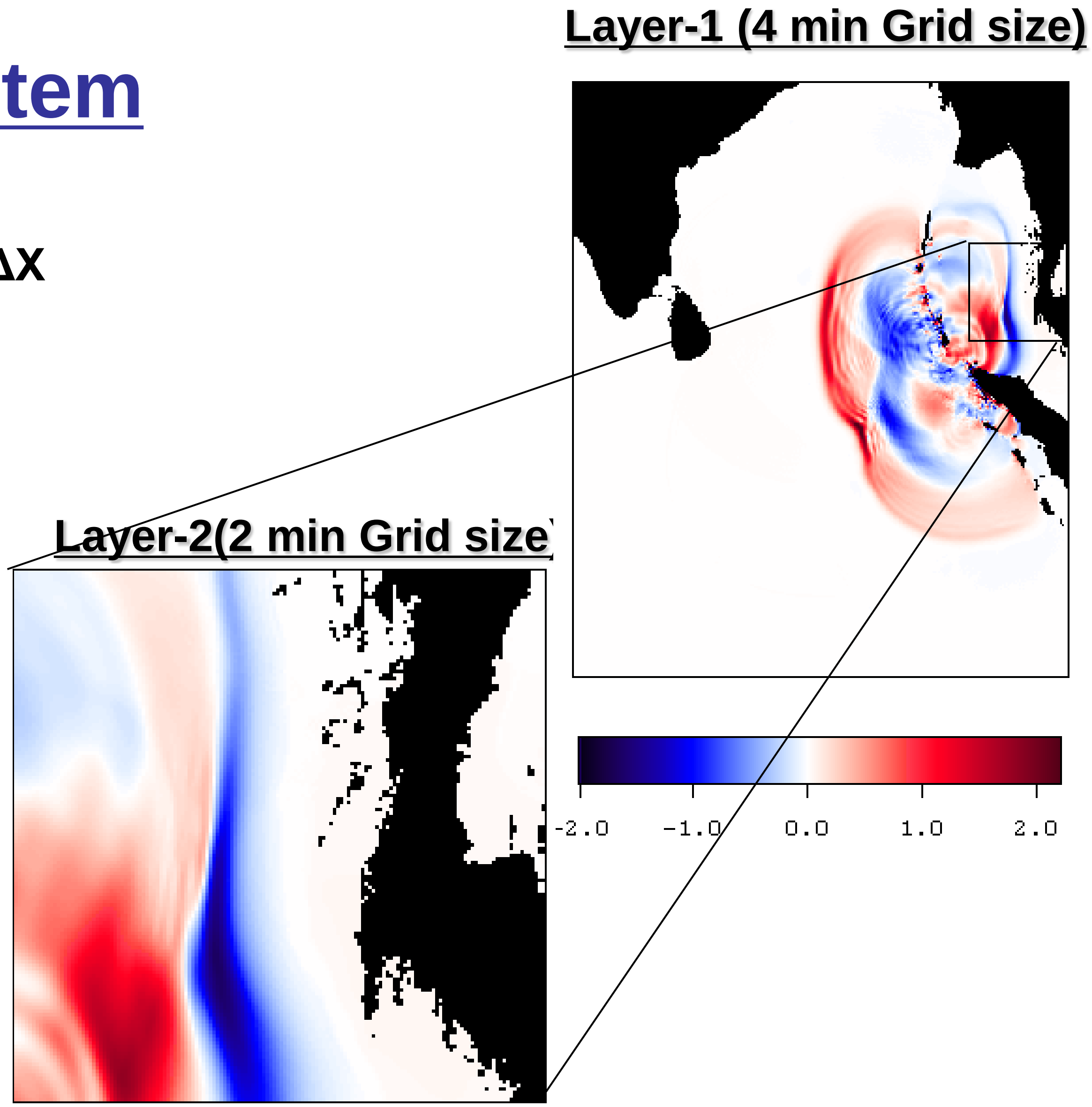


Nesting model

Nesting grid system

Nesting Grid system

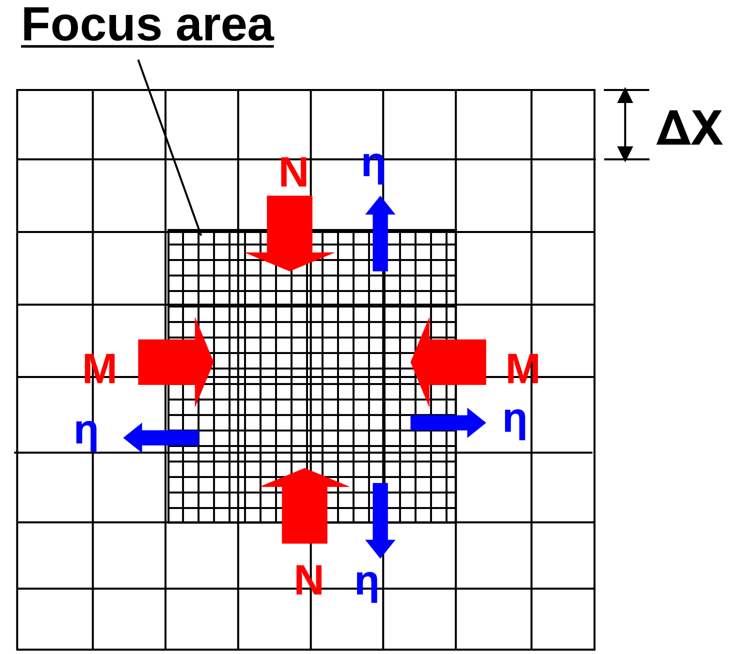


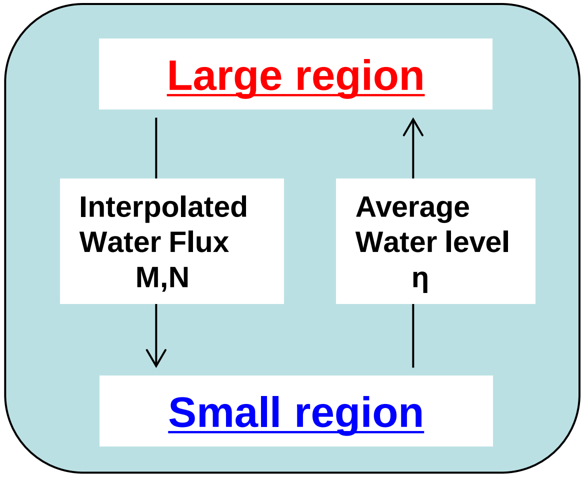


It is better to use detailed grid mesh. But it takes time to compute the detailed grid mesh.

So we use changing grid systems.

Nesting grid system



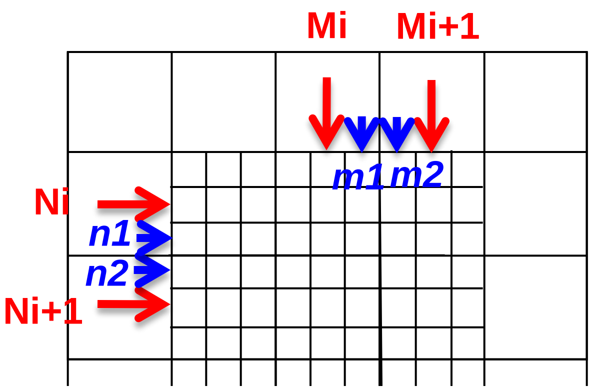


Small region receives interpolated water flux M,N from large region.

Large region receives averaged water level η from small region.

Nesting grid system

(1) Large region to Small region

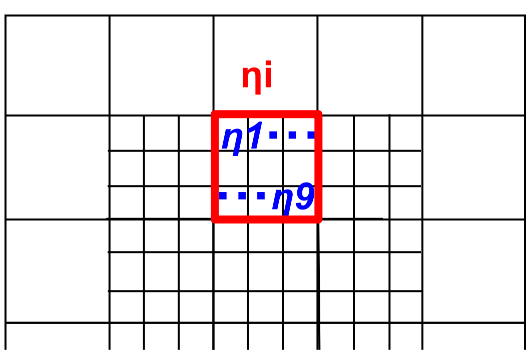


【Linear interpolation】

$$m1 = (2 \times Mi + Mi{+}1)/3$$

$$m2 = (Mi + 2 \times Mi{+}1)/3$$

(2) Large region to Small region



【Averaged water level】

$$\eta i = (\eta_1 + \eta_2 + \eta_3 + \cdots + \eta_n)/n$$

Algorism

Continuity equation of small region

↓

Insert the averaged η from small to Large region

↓

Momentum equation of Large region

↓

Insert the interpolated M and N from Large to small region

Exercise

Tsunami simulation

Original Fortran code:

TUNAMI (Tohoku University's Numerical Analysis Model for Investigation)

Governing equations:

Linear / Nonlinear long-wave / Dispersive wave model

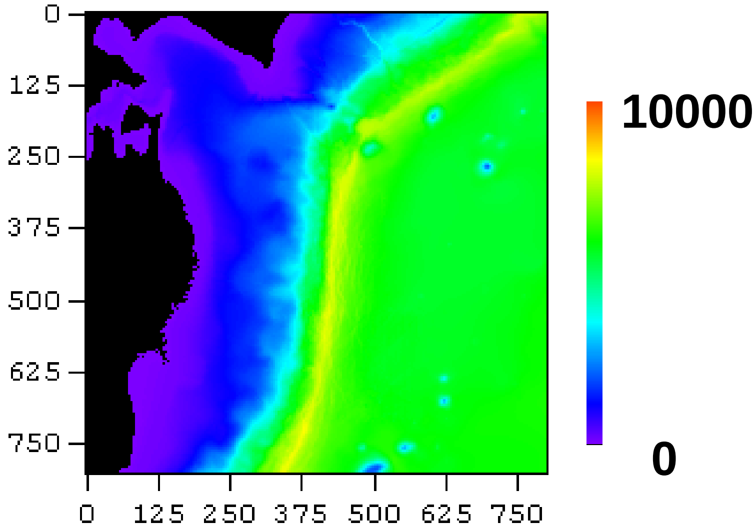
Coordinate systems:

Spherical (lat, long) / Cartesian (meter) coordinate system

Required data:

Bathymetry data (Digital elevation model)

→ **GEBCO**



Numerical condition

■ Temporal grid size: CFL condition

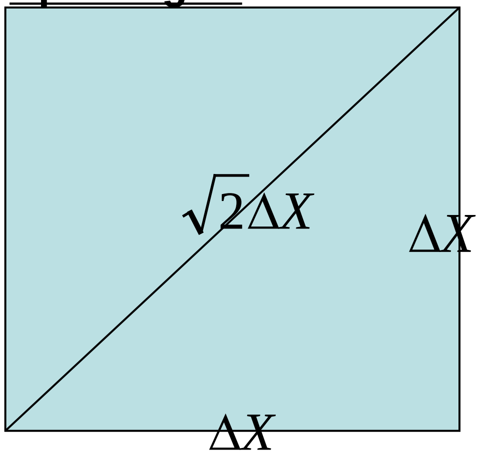
Velocity of numerical information

$$\sqrt{2}\frac{\Delta x}{\Delta t}$$

Velocity of tsunami wave

$$\sqrt{gh}$$

Spatial grid



$$\boxed{\sqrt{2}\frac{\Delta x}{\Delta t} > \sqrt{gh} \longrightarrow \Delta t < \Delta x\sqrt{\frac{2}{gh}}}$$

In nonlinear case, temporal grid size should be much smaller than cfl-condition

How to estimate required spatial grid size

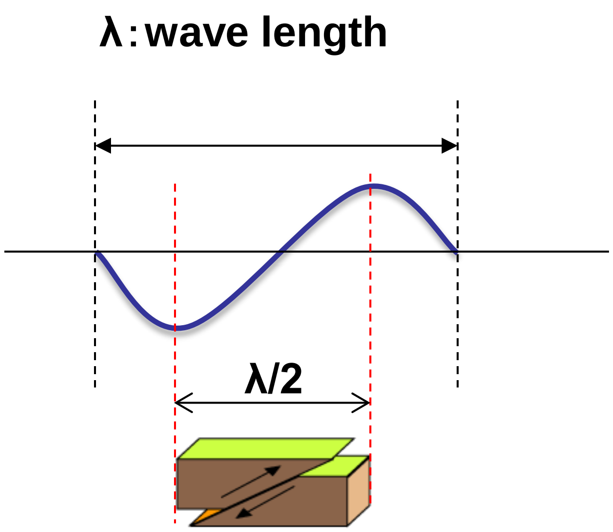
Numerical domain

$$\boxed{\Delta x < \lambda / a \quad \text{(a ≒ 20)}}$$

$$\lambda = \sqrt{gh} \times T$$

One-twentieth of wave length

$$\Rightarrow \Delta x_{max} = \sqrt{g h_{min}} \times T / a$$



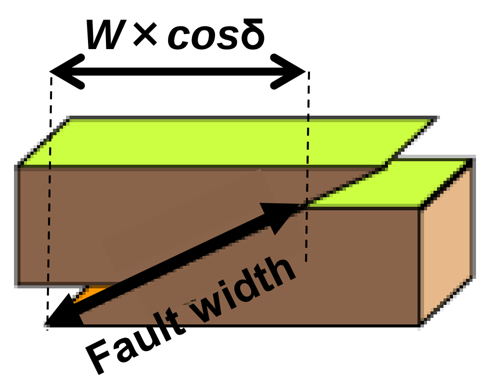
Δx is grid size, h is water depth, T is wave period

Here, we estimate wave period T from the condition at tsunami source (fault)

$$T = \lambda_0 / \sqrt{g h_0}$$

$$\lambda_0 = 2 \times \cos\delta \times W$$

$$\Rightarrow h_{min} = \frac{(a \Delta x_{max})^2 h_0}{\lambda_0^2}$$



λ_0 is wave length at tsunami source, h_0 is water depth at tsunami source, δ is dip of fault, W is fault width

Numerical condition

Numerical domain

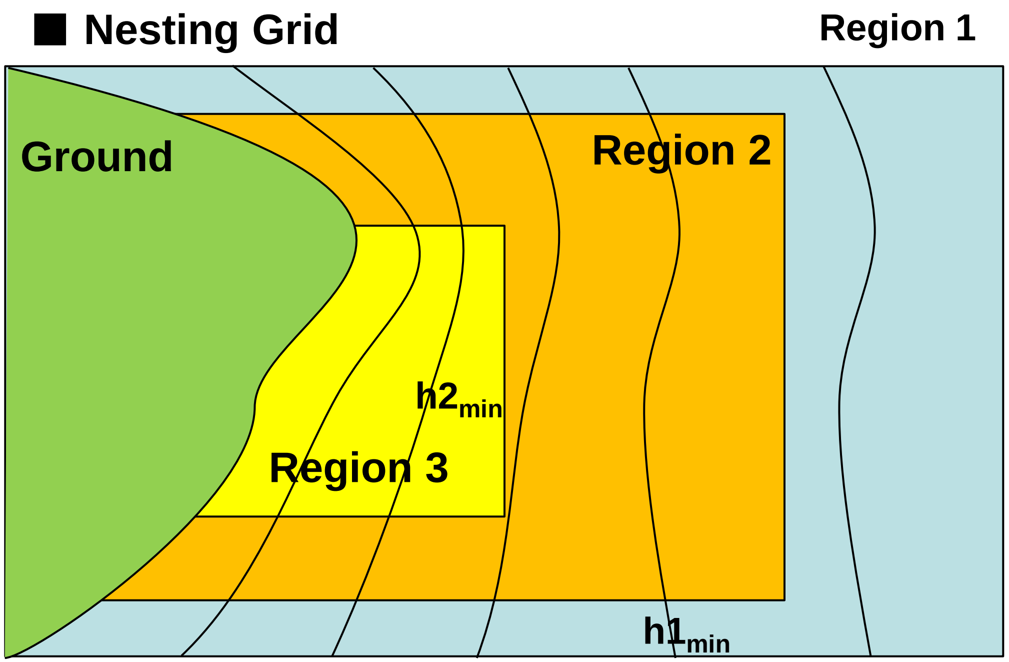
$$h_{\min} = \frac{(a\Delta x)^2 h_0}{\lambda_0^2}$$

Step 1 Set Δx in region 1

Step 2 Calculate h_{min} in region 1

Step 3 Determine area of region 2

■ **Nesting Grid**



■ **Example (W=50 km, δ=10deg, h0=5 km)**

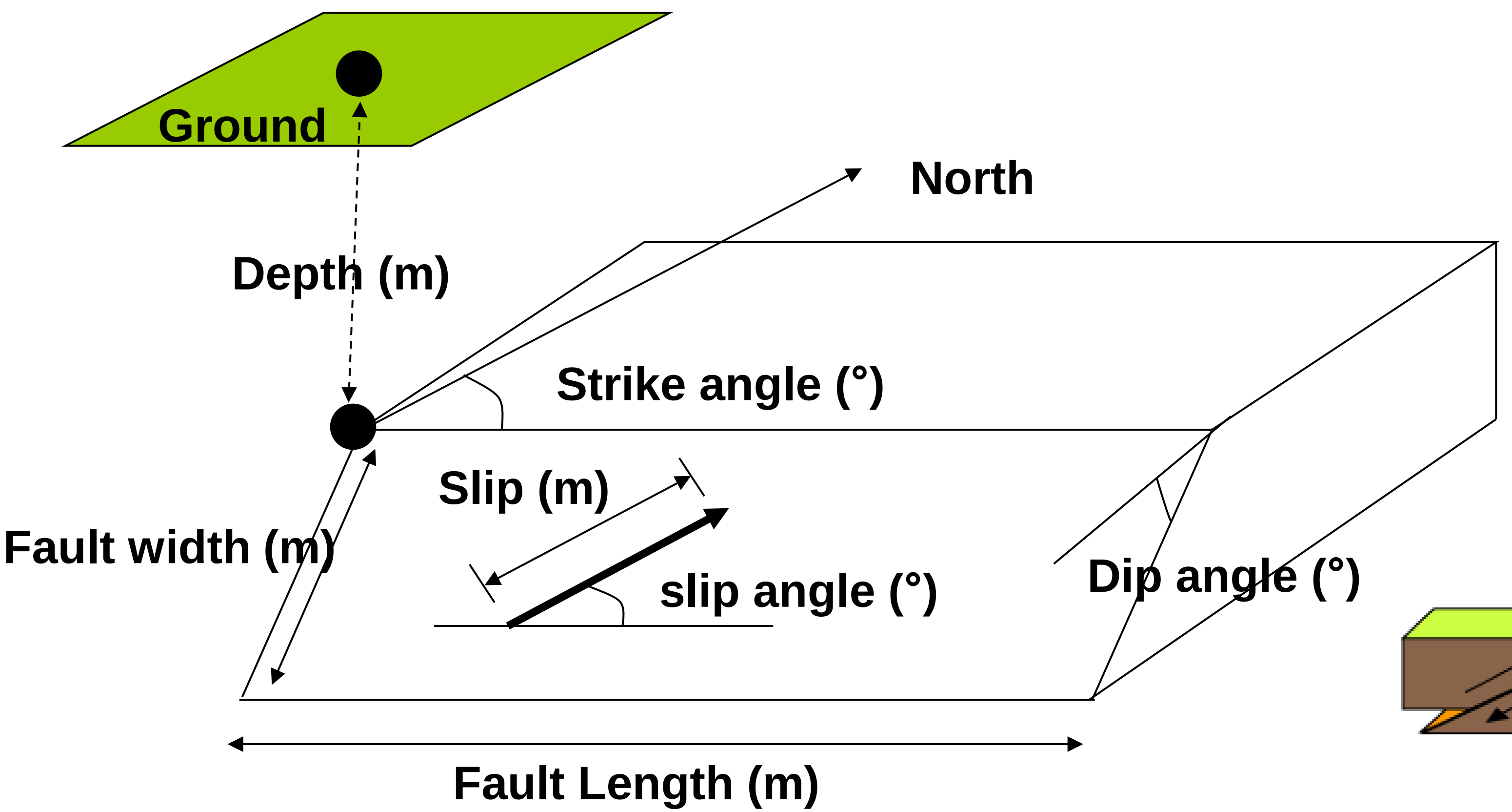
Region	*Δx*	h_{min}
1	1850	2823
2	617	314
3	206	35
4	69	4

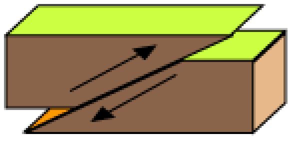
Nonlinear region (< h=50m)

It is better to extend more wide area

Tsunami source model

Manshinha & Smylie (1974)





Exercise

Software installation

1.GMT (The general mapping tools, http://gmt.soest.hawaii.edu/)

:Editor tool for Bathymetry data (GMT_basic_install.exe)

2. Ghostscript and GSView

:Viewer for postscript format (gs863w32full-gpl, gsv49w32.exe)

3. Gnuplot 4.6

Bathymetry data

- GEBCO (General Bathymetric Chart of the Oceans)

  Web page: http://www.gebco.net/
  Download: https://www.bodc.ac.uk/data/online_delivery/gebco/

- Grid size: 1 minute (2003 released and 2008 updated)
  30 second (2009 released)

- Format : NetCDF (Binary data)

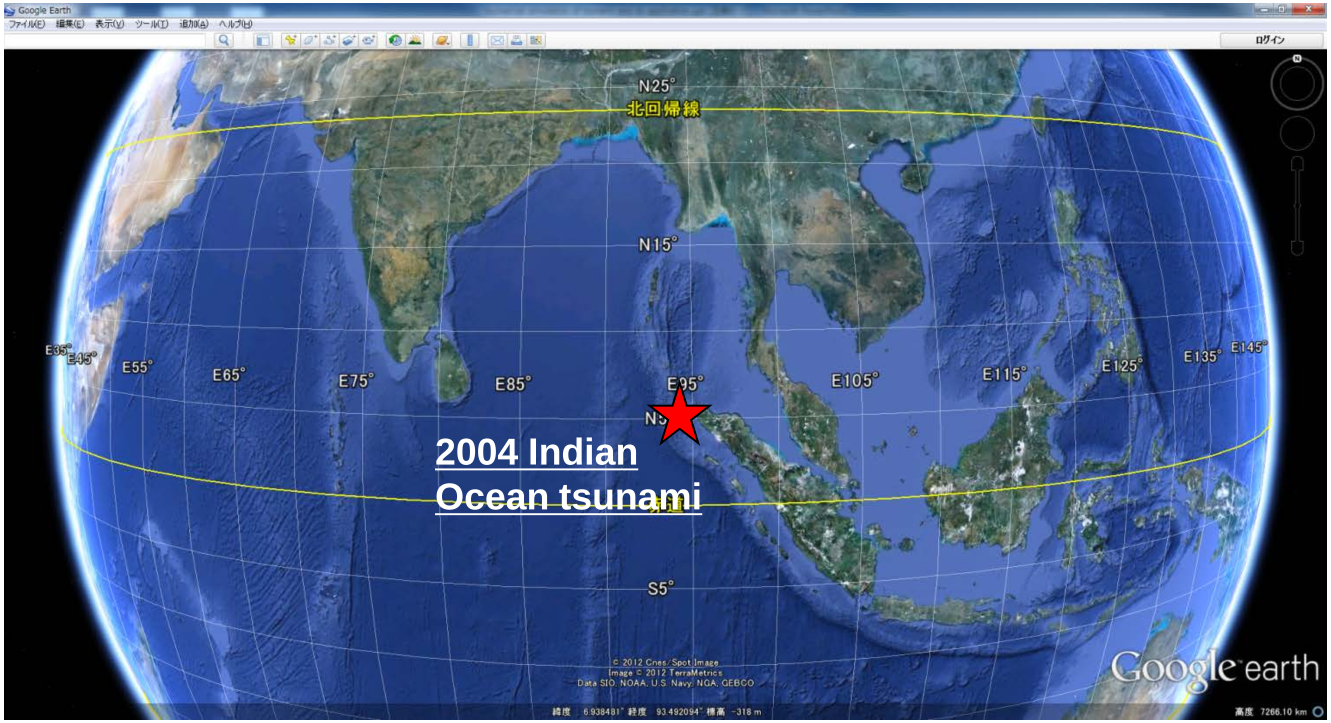
- Software : Grid display software (provided by GEBCO)
  GMT (http://gmt.soest.hawaii.edu/)

Exercise 1 (GMT)

Clip the bathymetry data around Indian Ocean

Requirement : Make two different grid size data => 4 min

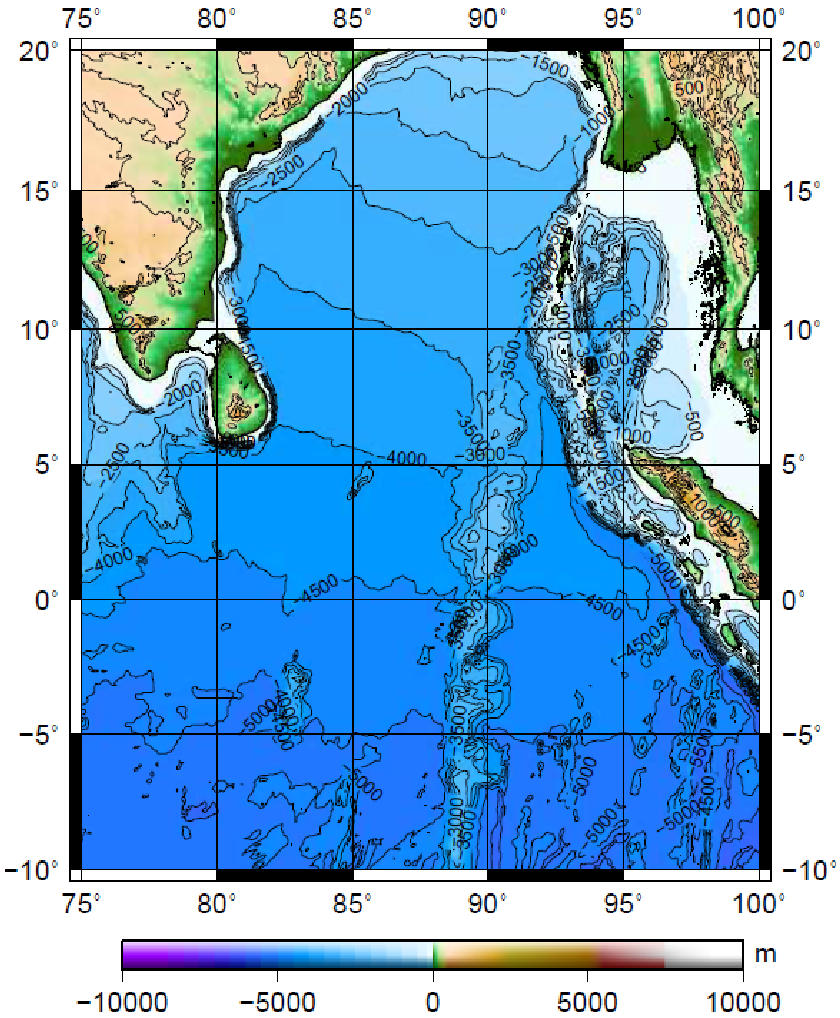
Data size should be Less than 400×400 Grid



(C) Google earth

Example (Indonesia)

4 min (75/100/-10/20)



4 min (your target area)

Chile

Myanmar

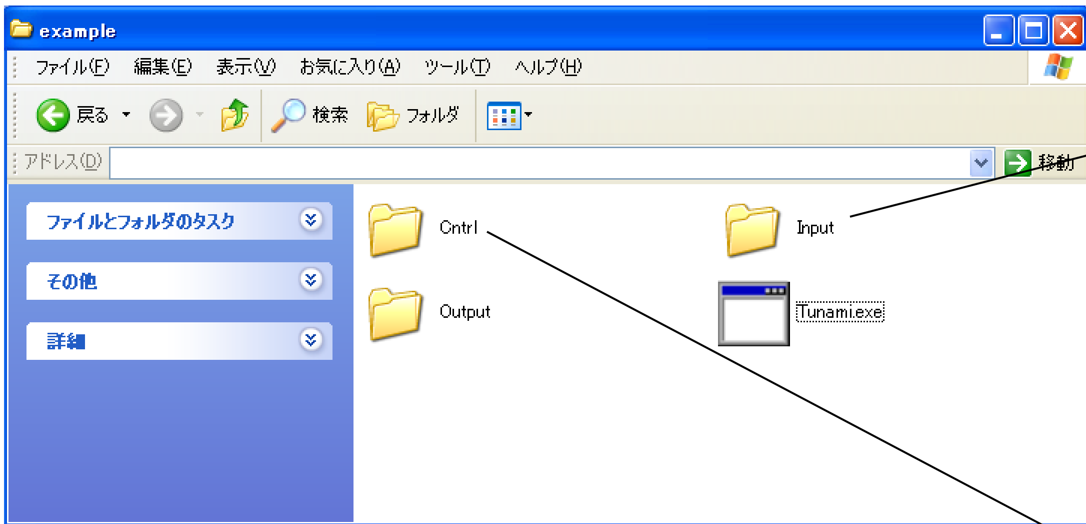
Papua New Guinea

Philippines

Etc..

TUNAMI MODEL

Example 1 (.. ¥TUNAMI MODEL¥example1(GAUSS))



Bathymetry files

Parameter files

(1) Binary data: Tunami-32bit.exe

(2) Cntrl folder: cntrl.fil , domains.cnt, faults.int, options.opt

(3) Input folder: bathymetry data (ASCII data)

(4) Output folder:

Flow

Put the bathymetry data into “input folder”

Set numerical condition in the following files

Change cntrl.fil ･･･ to set file name blow

Change ***.cnt ･･･ to set numerical condition

Change ***.int ･･･ to set tsunami source

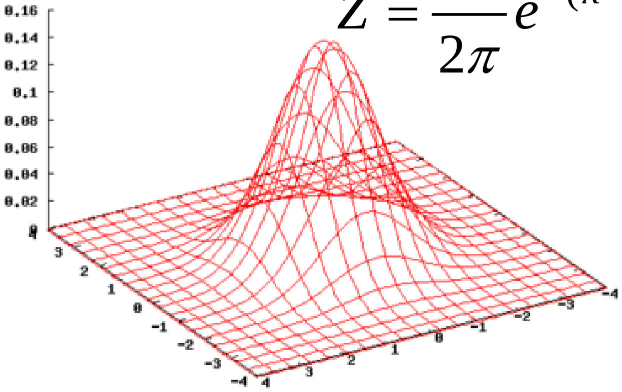
Change ***.opt ･･･ to do simple case test etc..

Click TUNAMI.exe

Exercise 2

Gaussian distribution

$$Z = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$$

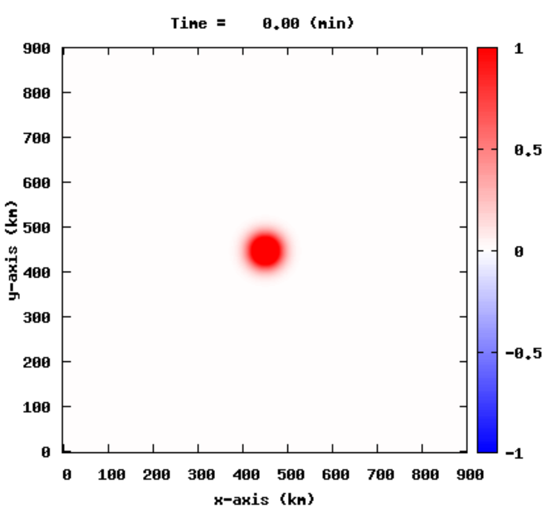


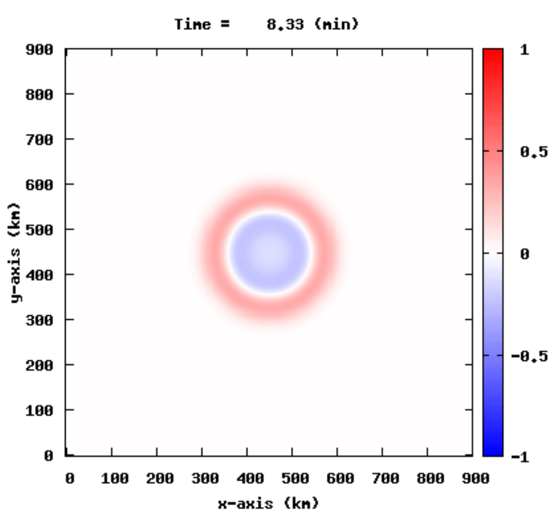
Initial elevation: Gaussian distribution

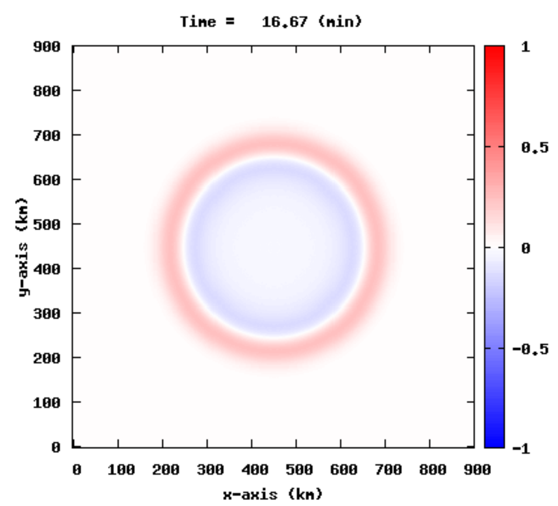
Water Depth: constant (=5000m)

Governing equation: Linear long-wave model

X,y-grid(300,300),dx=3000.0,dt=5.0







Exercise 3

Sine wave propagation in a channel

Initial elevation: Sine wave

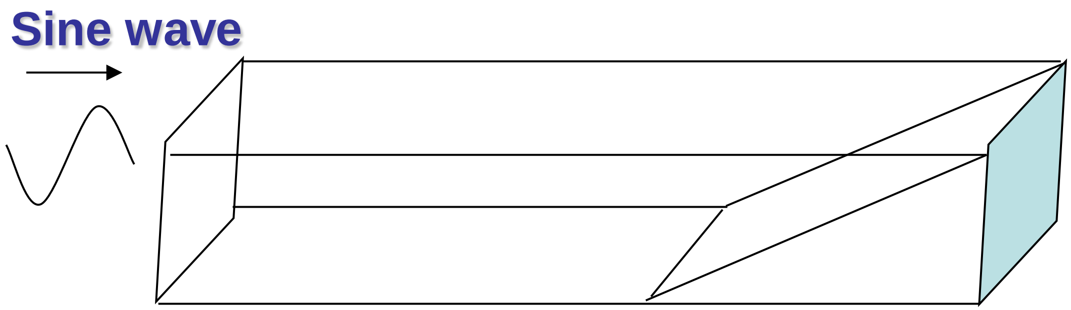
Water Depth: Channel.dat

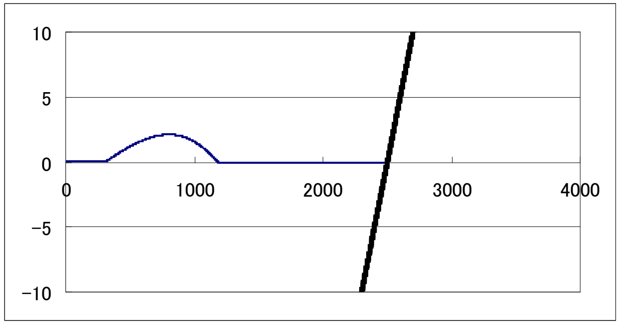
Governing equation:

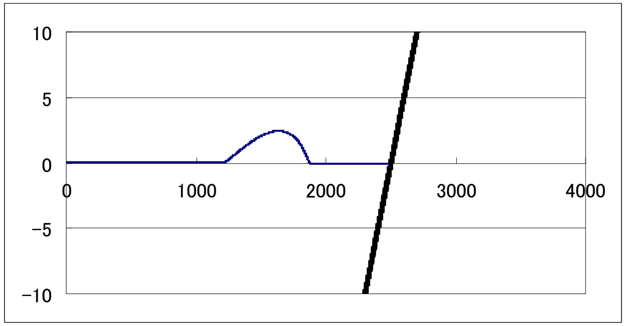
Nonlinear long-wave model

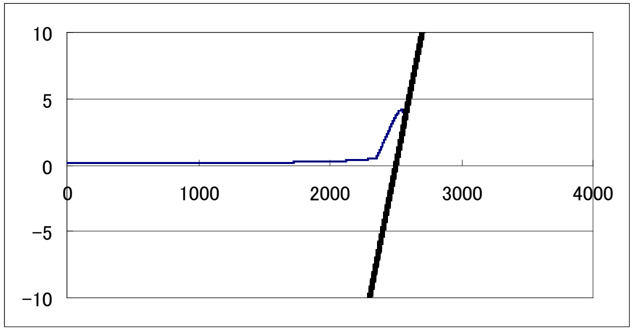
dx=10.0, dt=0.2, x-size=11, y-size=3000

A=5.0, Period=600.0, Duration=300.0





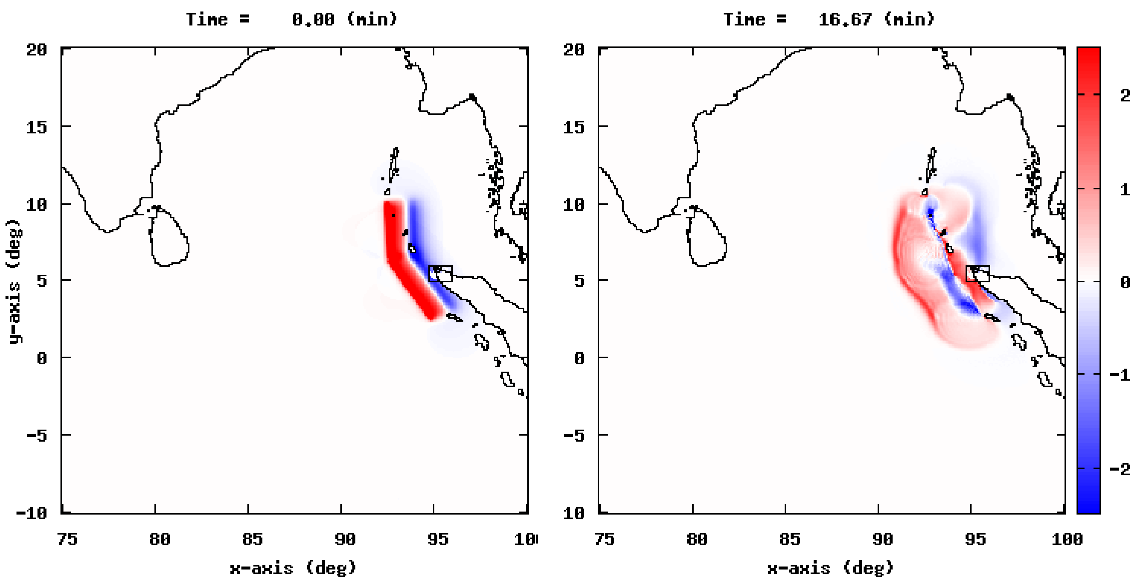




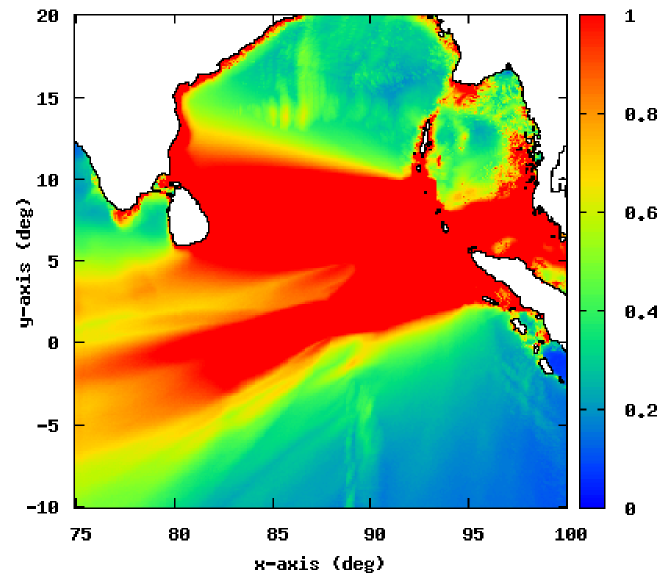
Exercise 4

Tsunami propagation in real bathymetry

Tsunami source: 2004 Indian Ocean tsunami



Maximum tsunami height



Fault Parameter of 2004 event

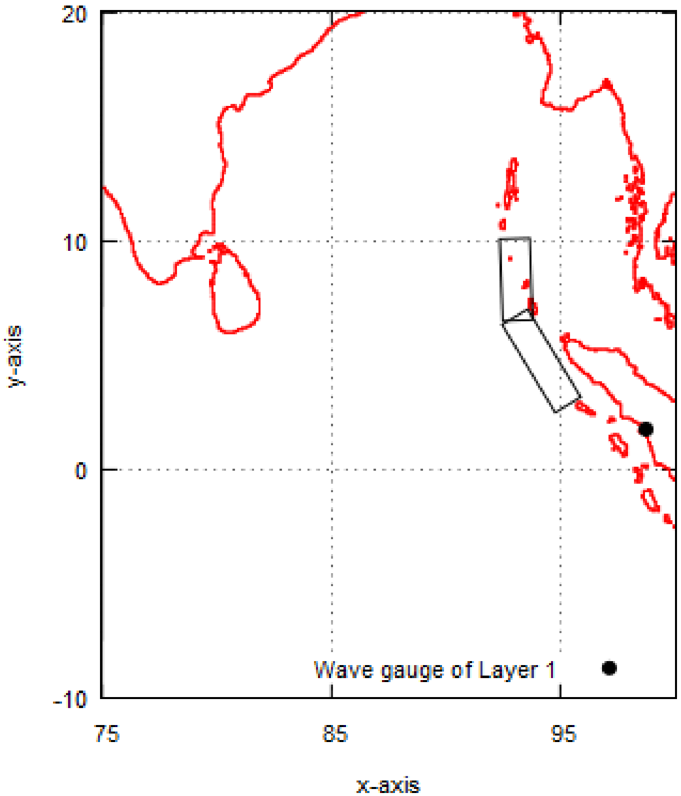
No	Slip (m)	Length (km)	Width (km)	Depth (km)	Strike (°)	Dip (°)	Slip angle (°)	Fault origin Lat (°)	Fault origin Long (°)
1	11	400	150	10	358	15	90	92.5	6.5
2	11	500	150	10	329	15	90	94.8	2.5

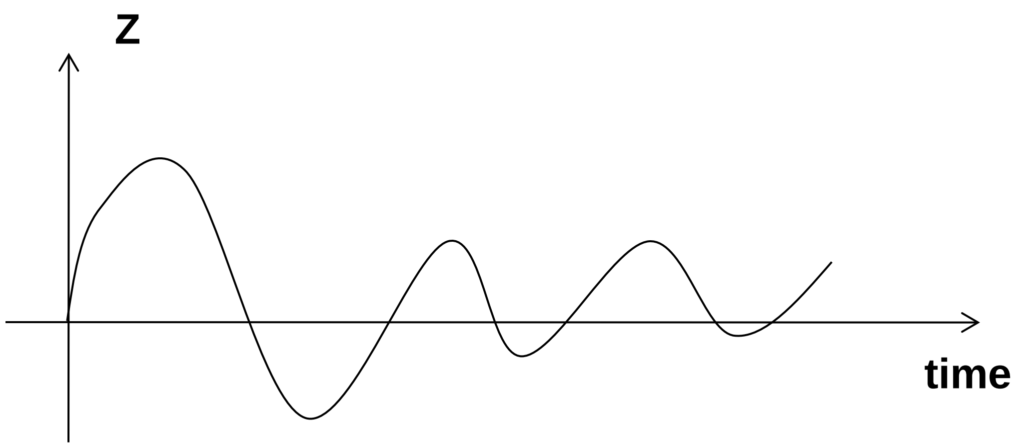
Exercise 4.1

Set tide-gauge data and make tide records

Tsunami source: 2004 Indian Ocean tsunami

Tide-record at SIBOLGA (98.76667 1.75) in 2004 Indian Ocean tsunami

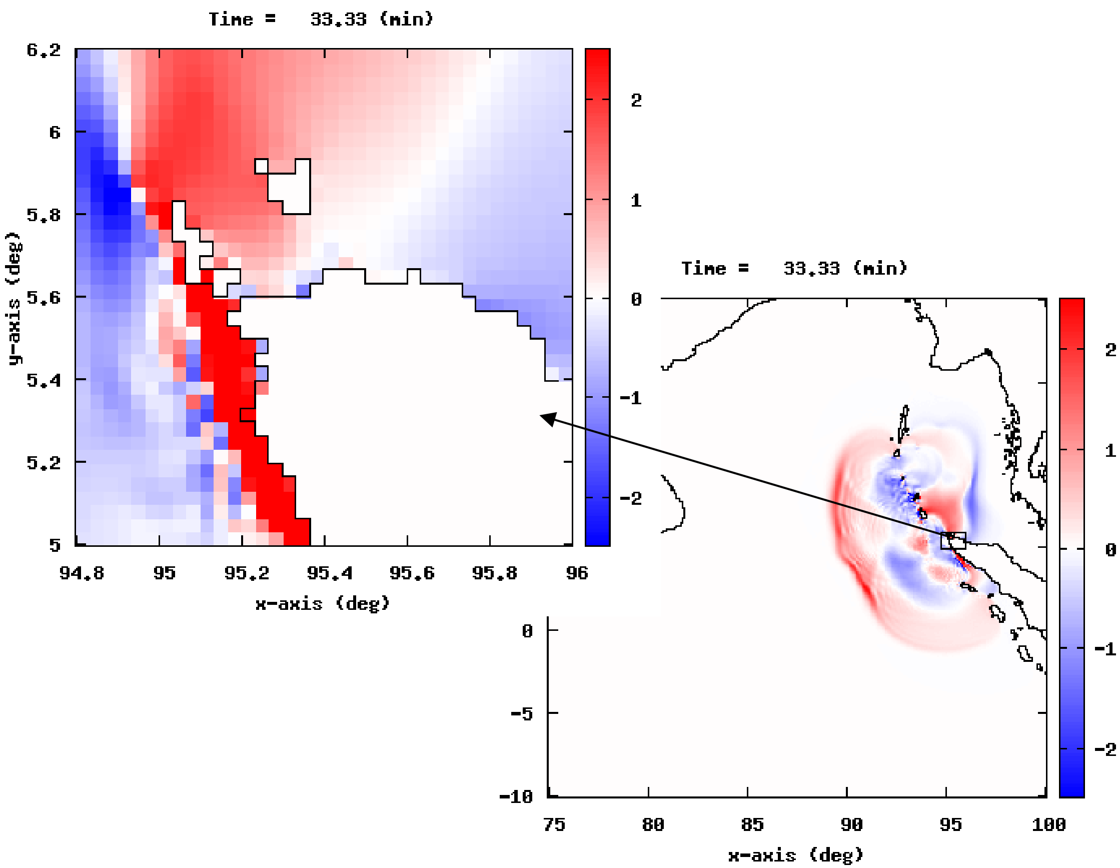


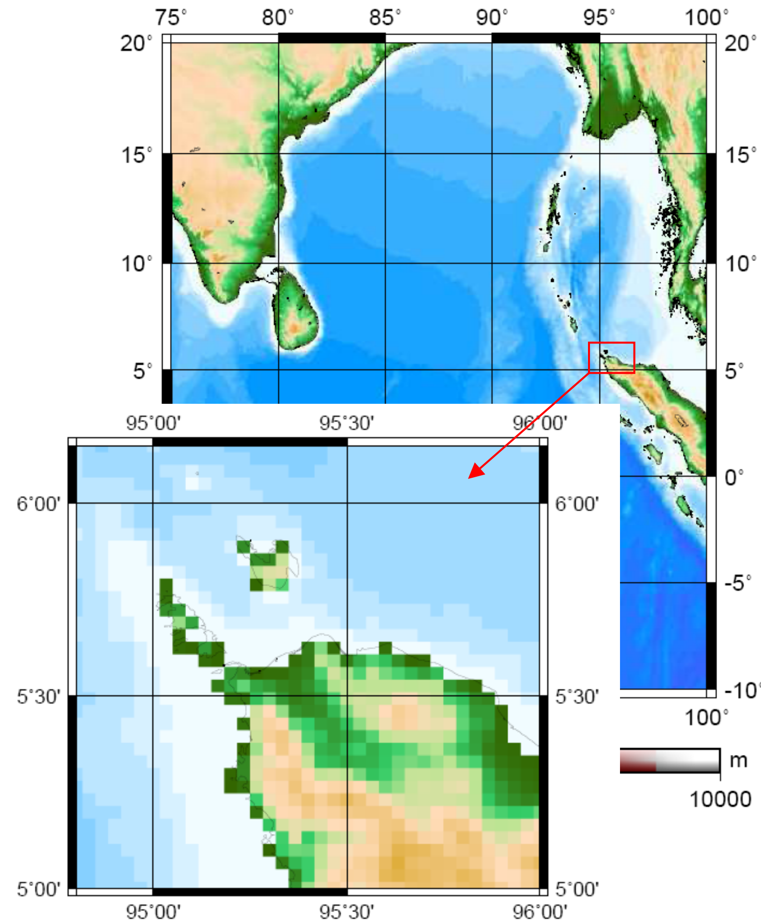


Exercise 5

Nesting in spherical coordinate system

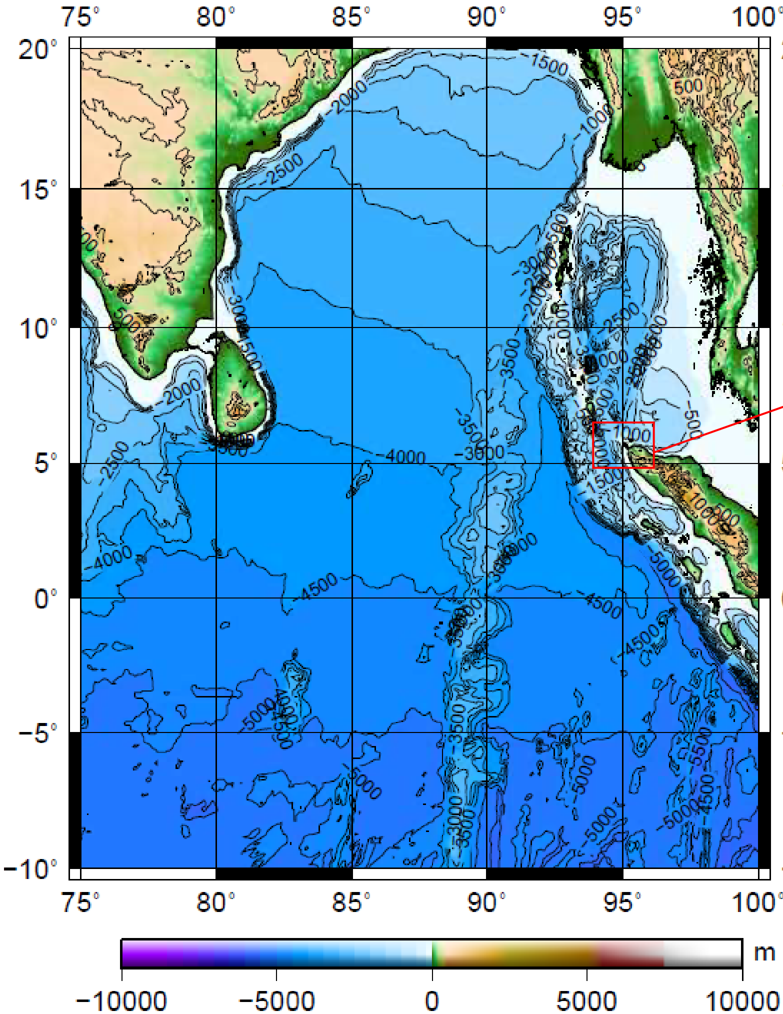
Tsunami source: 2004 Indian Ocean tsunami





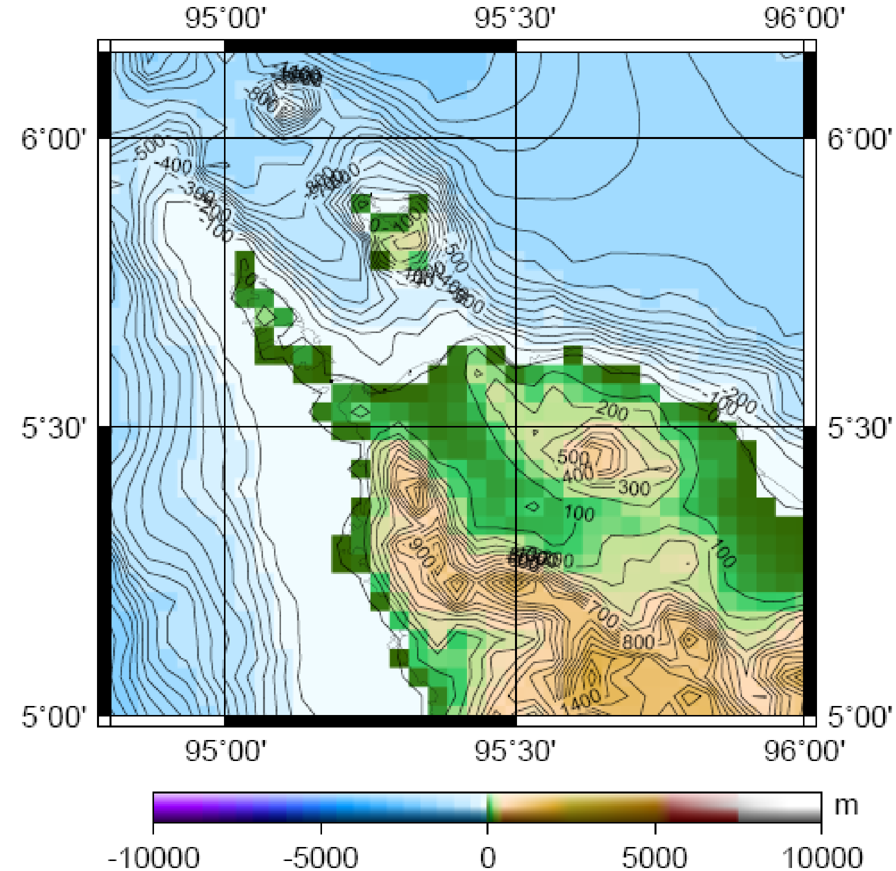
Clip bathymetry

4 min (75/100/-10/20)



2 min

$h_{min} \fallingdotseq 400$ m



Numerical condition

Numerical domain

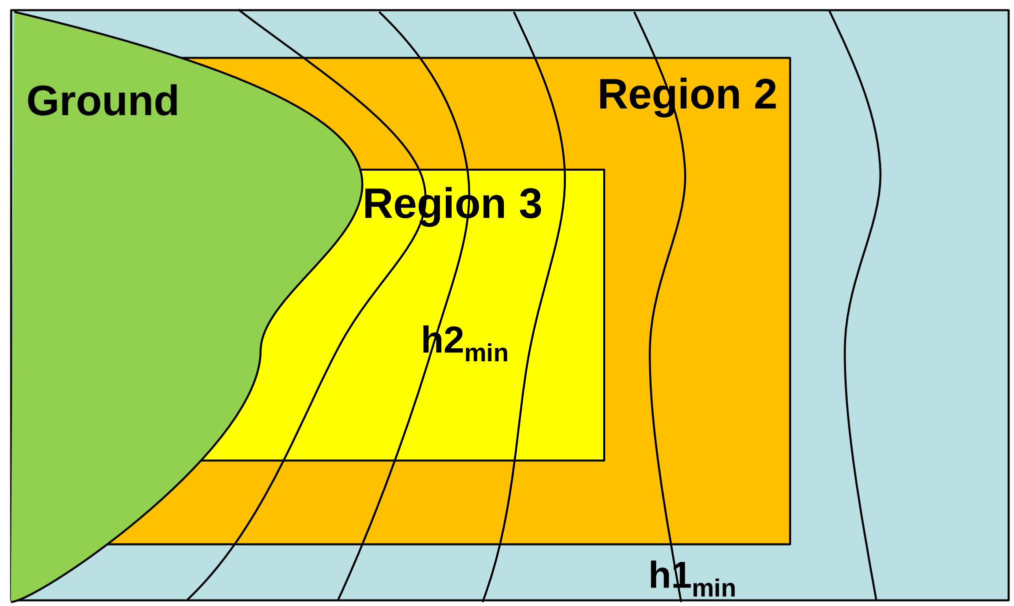
$$h_{\min} = \frac{(a\Delta x)^2 h_0}{\lambda_0^2}$$

Step 1 Set Δx in region 1

Step 2 Calculate h_{min} in region 1

Step 3 Determine area of region 2

■ **Nesting Grid**



■ **Example (W=50 km, δ=10deg, h0=5 km)**

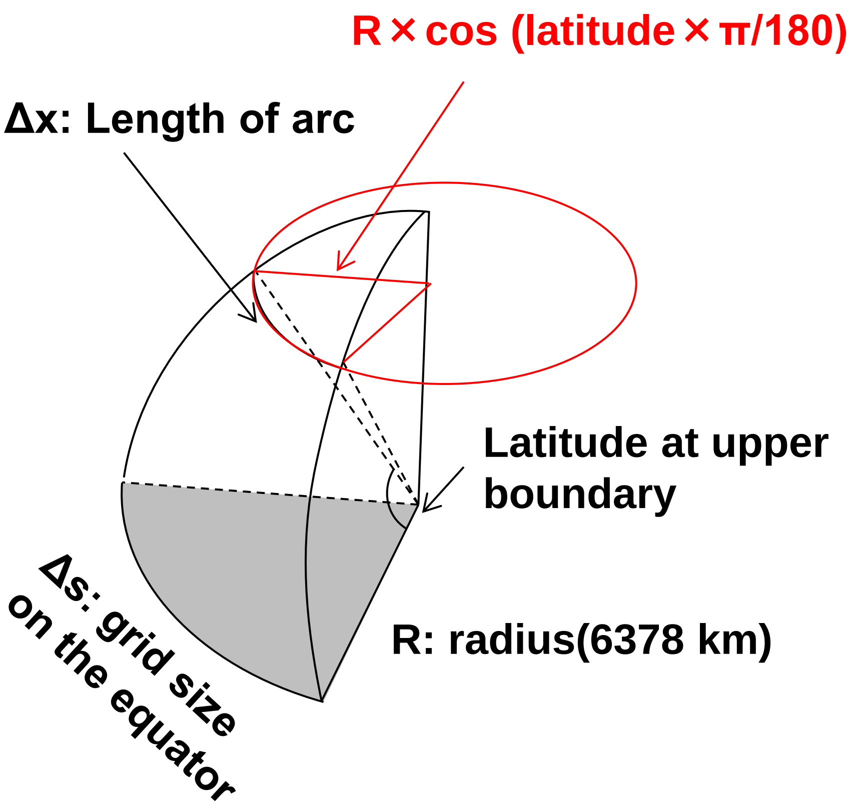
Region	*Δx*	h_{min}
1	1850	2823
2	617	314
3	206	35
4	69	4

Nonlinear region (< h=50m)

It is better to extend more wide area

Degree to meter

■ Simple estimation



Degree (minute) to meter

$$\Delta x = \Delta s \times R \times \cos(\text{latitude} \times \pi/180)$$

【Δx in meter】

Grid size/ Latitude(°)	0	5	10	20
1 min	1854	1847	1826	1743
2 min	3709	3695	3652	3485
3 min	5563	5542	5479	5228
4 min	7417	7389	7305	6971

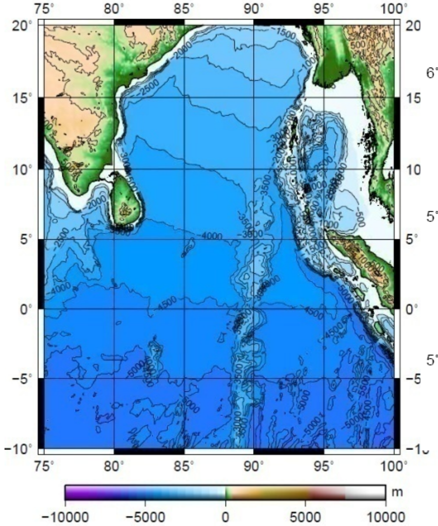
【Water depth: h_{min}】

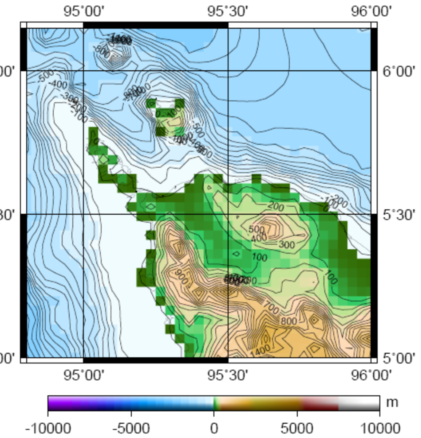
Indian Ocean tsunami (W=150km,dip=15deg, h0=1000 m)

Grid size/ Latitude(°)	0	5	10	20
1 min	~~25~~	~~24~~	~~24~~	~~22~~
2 min	98	98	95	87
3 min	221	219	214	195
4 min	393	390	381	347

Exercise 6

<u>Nesting in spherical coordinate system for your target area</u>

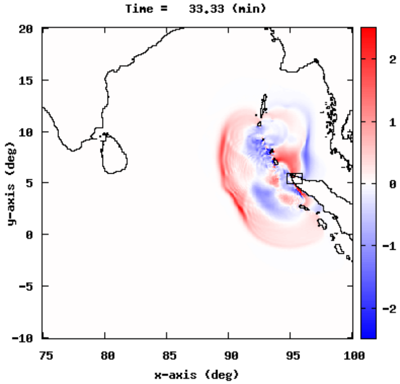




Step 1 Clip your target area

(4min and 2 min and 1 min)

Step 2 Set control files



Step3 Make animation and figures

(Water propagation and Maximum elevation)